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Transmission media – Characteristics

Mathematical models of multiplex signals

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NOTES

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2 In this Supplement, the expression "Administration" is used for conciseness to indicate both a telecommunication administration and a recognized operating agency.

Supplement No. 22

MATHEMATICAL MODELS OF MULTIPLEX SIGNALS

(Geneva, 1980; referred to in Recommendation G.223)

1 Introduction

Signals which represent the multiplex load on an FDM system can be defined in terms of the distribution of short-term power or instantaneous voltage. Values of these parameters are time dependent, and significant variation in the value of these parameters is to be expected even during the busy periods of successive days. Nevertheless, a means of determining even an "average" busy period distribution for the values of these parameters would be of great assistance in ensuring that planning margins were maintained as the system load was altered by the introduction of different types of traffic. To be of value, an estimate of the multiplex load distribution must be based on primary data that can be measured directly, or obtained from adequately accurate sources, so that the effects of any proposed changes in the data can be correctly incorporated; and the estimate must be produced in such a fashion that a direct measurement of the actual distribution can be performed and used to check the validity of the estimates. Methods of estimation which meets these requirements are described below.

Section 2 describes in general terms a process which may be used to calculate the probability density function (p.d.f.) of multiplex load short-term powers. In § 3, a mathematical process based on the work of Holbrook and Dixon is described. However these methods are now considered inexact for the purposes of modelling. The advent of high-speed digital computers has led to the development of the method described in § 4, by which means the instantaneous voltage distribution of the multiplex signal can best be calculated.

The methods described in § 4 are however not yet complete, in that they assume that the FDM load consists entirely of speech signals. Further studies to determine suitable descriptions for the distribution of instantaneous voltage due to signalling, supervisory tones, data etc. are being continued.

2 Method 1 – Probability density function of multiplex load short-term powers

2.1 The events occurring in an unidirectional channel of a "speech-band" circuit may be classified as follows:

1) main conversations *(sm)*;
2) minor (auxiliary) conversations *(sa)*;
3) signalling (numerical signals) *(zn)*;
4) supervisory tones (line signals) *(zl)*;
5) miscellaneous (e.g. data, echo) *(md, me)*;
6) idle *(i)*.

Measurements of signals in each of the classes during an adequate number of busy periods yield information from which can be obtained parameters which define the statistical properties of the channel load.

These parameters are:

1) the various mean overall activity factor, $\bar{\tau}$ (e.g. $\bar{\tau}_{sm}$, $\bar{\tau}_{sa}$, $\bar{\tau}_{zn}$, etc.) which define the fractions of time during which each of the classifications is producing active power in the average busy period;
2) the various mean $\bar{y}$ (e.g. $\bar{y}_{sm}$, $\bar{y}_{sa}$, $\bar{y}_{zn}$, etc.) and standard deviation σ (e.g. σ_{ysm}, σ_{ysa}, σ_{yzn}, etc.) of the p.d.f.s of the levels of active power for each classification.

2.2 The p.d.f. of the levels of short term (20 ms) active power being produced during an average busy period is calculated for each classification. Neglecting the possibility of certain fault conditions, these p.d.f.s can be regarded as mutually exclusive. Summing the probabilities of occurrence of a given power over all classifications therefore gives the total probability of occurrence of that power during the average busy period. This p.d.f. $p(Z_{uc})$ gives a sufficient description, statistically, of the unidirectional channel load.

$p(Z_{uc})$ can alternatively be obtained by direct measurement. However, the effect of changes in any of the classifications on the overall p.d.f. $p(Z_{uc})$, cannot then readily be determined.

2.3 One direction of transmission of an FDM group is formed from 12 unidirectional channels, each of which yields a p.d.f., $p(Z_{uc})$. If, as is usual, the signals in each channel are generated by statistically similar sources, the types of traffic carried by each channel will be in the same ratio, and it is then sufficient, for many purposes, to assume that each of the 12 p.d.f., $p(Z_{uc})$ can be represented by the same p.d.f., $p(Z_{tc})$ which is the p.d.f. of the power in the "typical channel" of the f.d.m. group.

The set of p.d.f., $p(Z_{tc})$ are not, of course, mutually exclusive, and the multiplex load due to the 12 channels is produced by convolution of this set of p.d.f. to form $p(Z_g)$. If pre-emphasis is used in transmission, it is merely necessary to multiply the power range of each p.d.f. by the appropriate pre-emphasis factor, f_p, prior to convolution.

The p.d.f. of multiplex load short-term powers for supergroup and larger group combinations, is obtained by further convolution of the group p.d.f.

2.4 The effects of traffic from nonspeech and nonconventional telephony channels can be incorporated as required. If introduced at the channel stage – e.g. MCVF, TASI – the p.d.f. $p(Z_{uc})$ will be modified accordingly prior to convolution. If introduced at the group, or higher stage, – e.g. wide-band data – one or more of the group power p.d.f. $p(Z_g)$, will be modified prior to convolution. The effects of group and supergroup limiters, and the inclusion of pilot, etc., tones may be incorporated as necessary.

The final p.d.f. obtained is the "unidirectional short-term multiplex power p.d.f.", $p(Z_{um})$. This p.d.f. is, of course, only applicable to the particular system and the particular type of signalling and traffic for which it was determined. For smaller systems, there will be some significant variations in the p.d.f. in different busy periods, but for larger systems, the variation can be expected to be relatively small.

2.5 The principles outlined above enable the distribution of the power of the multiplex load to be estimated. An equivalent procedure employing the voltage rather than the power statistics of the initial classifications of events yields an estimate of the distribution of the instantaneous amplitudes of the multiplex load. This is of importance in considering the probability of voltage overload of amplifiers in a given system (see for example § 4).

3 Method 2 – Equivalent peak power model based on the Holbrook and Dixon load rating theory (Source: Philips' Telecommunicatie Industrie BV)

In the theory of Holbrook and Dixon the maximum instantaneous power level of the multichannel signal is obtained by the addition of a multichannel equivalent signal power level L_m (the rms value exceeded with a probability of 1%, taking into account the distribution of active signal power levels and channel activity) and a multichannel peak factor $(\text{MPF})_n$.

3.1 *The number of active channels n*

The probability that during the busy hour any channel carries a signal is called the "activity factor", τ.

The probability that in an N-channel system exactly n channels are simultaneously active is given by the binomial p.d.f.

$$p(n) = \frac{N}{n!\,(N-n)!} \quad \tau^n (1-\tau)^{N-n}$$

If N is not too small, this binomial p.d.f. may be approximated by a normal p.d.f. with mean $\overline{n}$ and standard deviation σ_n where

$$\overline{n} = N\tau \text{ and } \sigma_n = \sqrt{N\tau(1-\tau)}$$

The number of activity channels which is exceeded with a probability of 1% is given by

$$n = N\tau + 2.33\sqrt{N\tau(1-\tau)}$$

3.2 *The n-channel equivalent signal power level L_m*

If the single-channel active signal power level $L_t = 10 \log_{10} W_t$ has a normal p.d.f. represented by $G(\overline{L_t}, \sigma_t)$, then the power W_t has a log-normal p.d.f. with mean value $\overline{W_t}$ and standard deviation σ_{wt} given by

$$\overline{W_t} = \exp\left[c\overline{L_t} + \tfrac{1}{2}(c\sigma_t)^2\right] \quad \text{and}$$

$$\sigma_{wt} = \overline{W_t}\sqrt{\exp(c\sigma_t)^2 - 1}$$

where

$c = 0.1 \ln 10 = 0.230$.

The long-term mean power is given by

$$W_0 = \overline{W_t} \cdot \tau$$

and the level of the long-term mean power by

$$L_0 = 10 \log_{10} W_0 = \overline{L_t} + 0.115\,\sigma_t^2 + 10 \log_{10} \tau.$$

If n is not too small, the power W_n of the sum of n active channels has a normal p.d.f. represented by $G(\overline{W}_n, \sigma_{wn})$

where

$$\overline{W}_n = n\overline{W}_t \text{ and } \sigma_{wn} = \sigma_{wt}\sqrt{n}$$

Hence the level of the power W_n which is exceeded with a probability of 1%, called the "n-channel equivalent signal power level, L_m", is given by

$$L_m = 10 \log_{10} (\overline{W}_n + 2.33\ \sigma_{wn}) = L_0 - 10 \log_{10} \tau + 10 \log_{10} \left\{ n + 2.33 \sqrt{n[\exp (0.23\ \sigma_t)^2 - 1]} \right\}$$

3.3 *Multichannel peak factor*

The multichannel peak factor $(\text{MPF})_n$ has been defined by Holbrook and Dixon as the ratio

$$(\text{MPF})_n = 20 \log_{10} \frac{\text{maximum instantaneous voltage exceeded with a probability } \varepsilon}{\text{rms voltage}}$$

for n active channels at constant volume. The value of probability ε to be used for satisfactory equipment design, etc., was not exactly determined, but was of the order of 10^{-4} or 10^{-5}. The $(\text{MPF})_n$ was determined empirically as a function of n, and a good fit for this function is the expression

$$\text{MPF}_n = 12.9 + [6/(1 + 0.07n)] \quad \text{dB.}$$

3.4 *Equivalent peak power P_{eq}*

The equivalent peak power is defined as the power of a sinusoid whose maximum amplitude equals the maximum instantaneous voltage of the signal from n active channels. Thus

$$P_{eq} = L_m + (\text{MPF})_n - 3 \text{ dBm0}$$

$$= L_0 - 10 \log_{10} \tau + 10 \log_{10} \left\{ n + 2.33 \sqrt{[n\,(2 \exp 0.23\ \sigma_t - 1)]} \right\} + 9.9 + 6/(1 + 0.07\ n) \quad \text{dBm0}$$

where

$$n = N\tau + 2.33 \sqrt{[N\tau\,(1 - \tau)]}.$$

Substituting the conventional parameter values

$L_0 = -15$ dBm0

$\sigma_t = 5.8$ dB

$\tau = 0.25$

the expression for P_{eq} becomes

$$P_{eq} = 10 \log_{10} (n + 5.17 \sqrt{n}) + 0.9 + 6/(1 + 0.07\ n) \quad \text{dBm0}$$

where

$$n = 0.25\ N + \sqrt{N}$$

This expression may also be written in the form

$$P_{eq} = -5.1 + 10 \log_{10} N + 10 \log_{10} \left\{ 1 + \frac{4 + 10.34 \sqrt{1 + 4/\sqrt{N}}}{\sqrt{N}} \right\} + \frac{6}{1 + 0.07\,(\sqrt{N} + N/4)} \quad \text{dBm0}$$

from which it can be seen that for large values of N, the expression for P_{eq} approaches the asymptote

$$\lim_{N \to \infty} P_{eq} = -5.1 + 10 \log_{10} N \quad \text{dBm0}$$

4 Method 3 – Models for instantaneous voltage distribution of FDM signals (Source: Bell-Northern Research and Philips' Telecommunicatie Industrie BV)

This model deals with the distribution of instantaneous telephone signal amplitudes on multichannel FDM systems. It is based on a knowledge of telephone signal amplitudes, single channel signal power levels, and activity factor.

4.1 *Telephone speech amplitude density function*

The probability distribution function for telephone speech voltage normalized with respect to the rms voltage is given by:

$$P(r) = \frac{s}{\Gamma(m)} (sr)^{m-1} \exp(-sr) \qquad 0 \leqslant r \leqslant \infty$$

where

$$r = \frac{|\text{ instantaneous voltage }|}{\text{rms voltage}} = \frac{v}{u}$$

$$s = \sqrt{m(m+1)}$$

4.2 *Single channel mean power levels*

The distribution of the mean power levels of the telephone signal while active can be represented by the expression:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\overline{x})^2}{2\sigma^2}\right]$$

where

$x = 20 \log_{10} u$

$\overline{x}$ is the mean value of x and σ is the standard deviation.

The distribution of u is log-normal:

$$g(u) = \frac{1}{u\sigma c\sqrt{2\pi}} \exp\left\{\frac{-(\ln u - c\overline{x})^2}{2\sigma^2}\right\}$$

where

$$c = \frac{1}{20} \ln 10 = 0.115$$

4.3 *Random channel amplitude density function*

The expressions in §§ 4.1 and 4.2 above can be combined by convolution to yield the distribution of active channel instantaneous voltage amplitudes.

$$h(v) = \int_{-\infty}^{+\infty} \frac{1}{r} P(r)\, g(u)\, dr$$

4.4 *Random channel amplitude density function*

The distribution given by § 4.3 above must be modified by the activity factor in order to produce the distribution of instantaneous voltage amplitudes on a random channel. Assuming a channel activity τ, and using a Dirac Delta function to model the activity factor, the distribution $h(v)$ from § 4.3 is modified thus:

$$p(v) = \tau \cdot h(v) + (1 - \tau)\,\delta(v)$$

where

$\delta(v) = 1$ for $v = 0$

$\delta(v) = 0$ for $v \neq 0$.

4.5 *Limiting*

In order to take account of the effects of limiting associated with the channel modulator (Recommendation G.232, § 8) the distribution from § 4.4 above must be modified. It can be assumed that any signal voltage in excess of the limiting level at the input to the channel modulator will appear at the channel modulator output at the limiting level. Thus the total probability of signal voltages in excess of the limiting level can be considered to be the probability of voltages occurring at the limiting level.

4.6 *Signals other than speech*

The distribution produced by § 4.5 above may be modified if necessary to include signals other than speech. By repeating processes in §§ 4.1 to 4.5 above with appropriate expressions for the instantaneous voltage distribution, distribution of mean power level, activity factor, etc., a distribution of instantaneous voltages other than speech will be produced. Provided that the occurrence of the various signals on a channel are mutually exclusive, the distributions may be combined by addition of the appropriate elements, to yield the distribution of instantaneous voltages due to all signals in a random channel.

4.7 *Multichannel amplitude density function*

By taking repeated convolutions of the distribution from § 4.6 above the multichannel amplitude density function can be obtained. This can be represented by the expression:

$$p_N(v) = p(v) * p(v) * p(v) \; ..... \; N \text{ times}$$

where

N represents the number of channels and

* denotes a convolution process.

Alternatively the combination may be performed using the characteristic function of $p(v)$.

4.8 *Example of calculations*

The procedure described above has been used to calculate the equivalent peak power level, assuming values for the parameters used in the various expressions. The calculations were made using the two sets of parameters given in Table 1.

TABLE 1

Values assumed for the various parameters

Parameter	m	$\bar{x}$	σ	τ	Limiting
Value (a)	0.2	– 12.9	5.8	0.25	+ 10 dBm0
(b)	1	– 15.1	6.4	0.35	+ 10 dBm0

The equivalent peak power level P_{eq} for an overload probability of $\varepsilon = 10^{-5}$ can be expressed as a function of the number of system channels N for the two sets (a) and (b) of parameters given in Table 2.

TABLE 2

N	1	2	12	24	36	48	60	120	300	600	900	1800	2700	10800
P_{eq} (a)	7.0	9.3	12.7	13.7	14.6	15.3	15.9	17.7	20.4	22.8	24.3	26.9	28.6	34.4
P_{eq} (b)	7.0	8.3	11.7	13.1	14.1	14.8	15.4	17.4	20.4	23.0	24.6	27.5	29.1	35.1

The difference P_{eq}(a) − P_{eq}(b) varies from 1.0 dB (N = 12) to −0.7 dB (N = 10 800). Analytical expressions may be fitted to the table values with a correlation of r = 0.9999 and r = 0.9998 respectively:

$$\text{parameter set (a): } P_{eq} = 13 \left[10^{-0.18\,(\log_{10} N)^{1.8}} \right] - 6.0 + 10 \log_{10} N \quad \text{dBm0}$$

$$\text{parameter set (b): } P_{eq} = 12.3 \left[10^{-0.264\,(\log_{10} N)^{1.53}} \right] - 5.3 + 10 \log_{10} N \quad \text{dBm0}$$

The last two terms in each of these expressions represent the limiting value of P_{eq} for $N \rightarrow \infty$.

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