Chaos Approach to Spectrum Management

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Abstract

Recently, chaos theory has developed a new area that is relevant to future radio spectrum management. Through the development of applications of chaos theory to communications systems, it turns out that a multi-channel chaotic spectrum spreading can be seen as a fusion of the CDMA technologies and the OFDM technologies, namely, a fusion of the two important communications technologies to attain high spectrum efficiency. Namely, CDMA and OFDM can be naturally unified into a single chaos communications technologies sharing the same radio spectrum with high spectrum efficiency. One important implication of this chaotic communications approach to future spectrum management is that we can monitor the spectrum usage and a new transmitter of potentially many different types of CDMA/OFDM mixing method in a same frequency band can be added with few degradation if it exploits chaotic signal modulation technique whose spectrum spreading signals are orthogonal to existing mixed signals. The theoretical basis of the effect of chaos can be explained by our recently proposed "Chaotic Analysis" which is to expand arbitrary given signals in terms of basis functions derived from complete orthonormal functions associated with Lebesgue spectrum of underlying chaotic dynamical system, which means that chaotic analysis can be another view of Fourier analysis, as the fundamental basis of spectrum management. We discuss the impact of chaos technologies to spectrum management from the above viewpoint and its relation to realization of market efficiency for future potential spectrum trading. We conclude our recommendation that a sufficient number of spectrum sharing technologies must be considered *before* putting spectrum on the market.In my presentation, we also present a model for spectrum licensing model with reselling incentives to give a suitable unpredictability of services and strategy of network operators and MVNOs.

1 Introduction of Chaos Approach

Spectrum spreading with chaotic sequences are concerned with new direction of key technology of spread spectrum sharing [2, 5, 9, 10, 13]. One of the remarkable features of chaotic spreading sequences is that there are potentially infinitely many different sequences that can form a unique ensemble according to the ergodic invariant measure. Furthermore, we can perform an exact statistical analysis for system performance of communication systems based on such chaotic spreading sequences by the ergodic principle[18]. In code-division multiple-access (CDMA) communication systems, orthogonal property between spreading sequences is an essential property for spectrum sharing. There has already been an extensive study about correlation properties of linear feedback shift register sequences such as m-sequences, Gold sequences and Kasami sequences[11]. The spectrum efficiency of chaotic spreading can be analyzed by "chaotic analysis" [16] which expands any given signal in Hilbert space in terms of complete orthogonal elements, which represent a chaotic element of signal. The following is the outline of the present paper. In Section 2, we shortly review the basis of our chaos approach. In Section 3, we review a chaotic analysis which gives another measure of radio spectrum. In Section 4, we provide an analysis of correlation properties of two signals which will be the basis of SNR analysis of Section 5. In Section 6, we review the core part of code design of our chaos approach to construct infinitely many modulations to share the same frequency spectrum band. Another implication of chaos analysis is related to constructing martingale fluctuation for market efficiency of spectrum trading. In Section 7, we discuss this matter and the relation between market efficiency of spectrum trading driven by chaotic fluctuation with a martingale property and chaotic spectrum spreading. Concluding remarks are given in the final section.

2 Ergodic Dynamical Systems: The basis of Chaos Approach

We consider a dynamical system $X_{n+1} = F(X_n)$ given by a mapping function $F : \Omega \to \Omega$ on a state space Ω. Let us assume that a mapping F is ergodic with respect to an invariant probability measure $d\mu(x) = \rho(x)dx$ with $\rho(x)$ being a continuous density function $\rho : \Omega \to R$. This means that the measure $d\mu(x) = \rho(x)dx$ is invariant under time evolution of F and, furthermore, is absolutely continuous with respect to the Lebesgue measure on Ω. In this case, according to the Birkhoff Ergodic Theorem [18], for any integrable function $A(x)$, an average over time $\lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} A(X_i)$ equals the average over the state space $\int_\Omega A(x)\rho(x)dx$ for almost every X_0 with respect to the probability measure $\rho(x)dx$. This means that an average value of a limit sequence

$$A(X_1), A(X_2), A(X_3), \cdots, A(X_N)\cdots \tag{1}$$

can be computed by the following ergodic equality

$$\overline{A} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} A(X_i) = \langle A\rangle \equiv \int_\Omega A(x)\rho(x)dx \tag{2}$$

for almost all X_0. Since, the ergodic invariant measure $\rho(x)dx$ is realized for almost all initial conditions on Ω, we can consider $\langle A\rangle$ as an ensemble average of A, where the initial conditions of each sample are distributed according to the invariant probability measure $\rho(x)dx$. Thus, we can evaluate an ensemble average for successively and finitely generated observable $A(X_i) = A(X_i)$ [4]. Suppose N successive observations, $A(X_i), \rho(X_i), \quad i = 1, \cdots, N$, of quantities $A(x)$ and a density function $\rho(x)$ have been stored. Note that

$$\langle\langle [\frac{1}{N}\sum_{i=0}^{N-1} A(X_i) - \langle A\rangle]\rangle\rangle = 0, \tag{3}$$

where the expectation of A denoted by $\langle\langle A\rangle\rangle$ means an ensemble average with respect to the initial conditions X_0 with a sampling measure $\rho(x)dx$. It is more important to consider *ensemble-average fluctuation* given by the expected value of the square of the error

$$\sigma(N) \equiv \langle\langle [\frac{1}{N}\sum_{i=0}^{N-1} A(X_i) - \langle A\rangle]^2\rangle\rangle. \tag{4}$$

As shown in Appendix A of Ref. [4], this expected value $\sigma(N)$ is explicitly written in terms of sum of two-point correlation functions of $A_i = A(X_i)$ as follows:

$$\sigma(N) = \frac{1}{N}\{\langle A^2\rangle - \langle A\rangle^2\} + \frac{2}{N}\sum_{j=1}^{N}(1-\frac{j}{N})\{\langle A_0 A_j\rangle - \langle A\rangle^2\}. \quad (5)$$

$\sigma(N)$ is composed of *the statistical variance term*

$$\sigma_s(N) \equiv \frac{1}{N}\{\langle A^2\rangle - \langle A\rangle^2\}, \quad (6)$$

which purely depends on the form of the integrand A and *the dynamical correlation term*

$$\sigma_d(N) \equiv \frac{2}{N}\sum_{j=1}^{N}(1-\frac{j}{N})\{\langle A_0 A_j\rangle - \langle A\rangle^2\}, \quad (7)$$

which *depends* on the chaotic dynamical systems $X_{n+1} = F(X_n)$ utilized as random-number generators and the integrand functions $A(x)$.

3 Chaotic Analysis

In this section, we explain a chaotic analysis by using Lebesgue spectrum's orthonormal functions and a basic tool of computing two-point correlation function of variables generated by ergodic dynamical systems is given. First, we consider a dynamical observable $\phi(x) \in L_2$. The L_2 space is a Hilbert space with a scalar product defined by

$$\langle u, v\rangle \equiv \int_\Omega u(x)v(x)\rho(x)dx. \quad (8)$$

Such a Hilbert space has finite or countably infinite orthonormal basis functions $\{\phi_j\}$ of L_2 which satisfy the following relation

$$\langle \phi_i, \phi_j\rangle = \delta_{i,j}. \quad (9)$$

When such orthonormal basis is *complete*, then for any $\phi(x) \in L_2(\Omega)$, $\phi(x)$ can be uniquely expanded as follows.

$$\phi(x) = \sum_{j=0}^{\infty} a_j \phi_j(x). \quad (10)$$

Chaotic Analysis is nothing but an expansion of a given signal in Eq.10 by the orthogonal signals $\phi_j(x)$ related to chaotic dynamical systems. Here, Lebesgue spectrum of ergodic dynamical systems is introduced by an orthonormal basis $\{\phi_{\lambda,j}\}_{\lambda\in\Lambda,j\in\Xi}$ for the Hilbert space L_2 having a special composite property [1]. Here, λ labels the classes splitting the orthonormal basis and j which is an element of the set Ξ of non-negative integers labels the functions within each class. Each class has infinitely many functions and the cardinality of Λ is uniquely determined by the underlying ergodic dynamical system $X_{n+1} = F(X_n)$. If the cardinality of Λ is infinite, the corresponding Lebesgue spectrum is called *infinite Lebesgue spectrum*. The special property of Lebesgue spectrum is given by the following composite property

$$\phi_{\lambda,j} \circ F(x) = \phi_{\lambda,j+1}(x), \quad \text{for} \quad \forall\lambda \in \Lambda, \forall j \in \Xi. \tag{11}$$

This means that if $\phi_{\lambda,0}$ is given, all the other basis functions $\{\phi_{\lambda,j}\}_{j\geq 1}$ can be generated from it simply by using compositions with power of F. By construction, each function is orthogonal both to every other function in the same class, and to every function in every other class. Furthermore, we can define the projected Hilbert space $L_2(\lambda)$ which corresponds to the class $\lambda \in \Lambda$ of the Lebesgue spectrum as follows. Let us consider a function $\phi_\lambda \in L_2$ given by

$$\phi_\lambda(x) = \sum_{j=0}^{\infty} a_{\lambda,j}\phi_{\lambda,j}(x), \tag{12}$$

where

$$\langle \phi_\lambda, \phi_\lambda \rangle = \sum_{j=0}^{\infty} |a_{\lambda,j}|^2 < \infty. \tag{13}$$

Such a set of functions $\{\phi_\lambda(x)\}$ characterize a special class of L_2 and we denote it $L_2(\lambda)$. $L_2(\lambda)$ can be seen as a projected Hilbert space of L_2. An important property of $L_2(\lambda)$ is again the composite property such that if $\phi(x) \in L_2(\lambda)$, then $\phi \circ F(x) \in L_2(\lambda)$. By the orthogonal property of the Lebesgue spectrum, each function of $L_2(\lambda)$ is orthogonal to every other function in every other projected Hilbert space $L_2(\lambda' \neq \lambda)$. Let us consider an orthogonal complement $M^\perp \in L_2$ of $M \equiv \bigoplus_{\lambda\in\Lambda} L_2(\lambda)$, where each function of $M^\perp$ is orthogonal to every function of $M = \bigoplus_{\lambda\in\Lambda} L_2(\lambda)$. In Appendix, such orthonormal bases related to ergodic dynamical systems, where the orthogonal complement $M^\perp$ corresponds to the set of constant functions will be constructed based on the well-known classical Chebyshev orthogonal polynomials. In general, the following relations

$$L_2 = M \bigoplus M^{\perp}, \quad M^{\perp\perp} = M \tag{14}$$

hold.

Thus, the orthogonal basis $\{\phi_{\lambda,j}\}_{\lambda\in\Lambda, j\in\Xi}$ together with a function $\phi_0 \in M^{\perp}$ forms a *complete* orthonormal basis of L_2; i.e., each function $\phi(x) \in L_2$ is uniquely expanded by

$$\phi(x) = a_0\phi_0(x) + \sum_{\lambda\in\Lambda} \phi_\lambda(x) = a_0\phi_0(x) + \sum_{\lambda\in\Lambda}\sum_{j=0}^{\infty} a_{\lambda,j}\phi_{\lambda,j}(x). \tag{15}$$

The example of orthonormal functions of L_2 is

$$\phi(x) = \sum_{j=1}^{\infty}(1/2)^j T_{p^j}(x), \tag{16}$$

which is well-known to be nowhere differentiable continuous functions in L_2 and T_m is a m-th order Chebyshev polynomial function. The fact that orthogonal basis may be nowhere differentiable continuous functions represents a difference between chaotic analysis and Fourier analysis, while their representation capacities of signals are the same as the class of functions in the Hilbert space.

4 Correlation Properties

Let us consider a normalized integrand $B(x) \in L_2$. By the above property of $L_2(\lambda)$, $B(x)$ has the following unique expansion:

$$B(x) = \sum_{\lambda\in\Lambda}\sum_{j=0}^{\infty} a_{\lambda,j}\phi_{\lambda,j}(x), \tag{17}$$

we can compute the l-shift correlation function $\langle B_0 B_l\rangle$ as follows:

$$\begin{aligned}\langle B_0 B_l\rangle &= \langle \sum_{\lambda\in\Lambda}\sum_{j=0}^{\infty} a_{\lambda,j}\phi_{\lambda,j}(x), \sum_{\lambda\in\Lambda}\sum_{j=0}^{\infty} a_{\lambda,j}\phi_{\lambda,j+l}(x)\rangle \\ &= \sum_{\lambda\in\Lambda}\sum_{m=l}^{\infty} a_{\lambda,m}a_{\lambda,m-l}.\end{aligned} \tag{18}$$

Here, we use the orthogonal property between every pair of functions in every different projected Hilbert spaces $L_2(\lambda)$ and $L_2(\lambda')$. This exact formula 18 of correlation functions can be exploited to give the optimal spreading sequence for asynchronous CDMA systems [13, 15].

5 SNR Analysis

Since spectrum efficiency is closely related to SNR (signal-to-noise-ratio) by the Shannon theory, we consider an ensemble average of SNR(signal-to-noise-ratio) for chaotic spreading sequences generated by ergodic dynamical systems with Lebesgue spectrum. Let us consider two different sequences

$$B(X_1), B(X_2), \cdots, B(X_N) \tag{19}$$

and

$$B(Y_1), B(Y_2), \cdots, B(Y_N) \tag{20}$$

generated by an ergodic dynamical system $x_{n+1} = F(x_n)$, where $X_1 \in \Omega$ and $Y_1 \in \Omega$ are independently chosen initial conditions and $B \in M = \bigoplus_{\lambda \in \Lambda} L_2(\lambda)$. Note that in this case, $B(x) \in M$ has the unique expansion (17) in terms of orthonormal functions with Lebesgue spectrum. Let us assume that the average power of sequences divided by the code length N has a constant value P_0:

$$\begin{aligned}\langle P_s \rangle = \langle \sum_{j=1}^{N} B^2(X_j) \rangle = N \int_{\Omega} B(x)^2 \rho(x) dx \\ = \{ \sum_{\lambda \in \Lambda} \sum_{m=0}^{\infty} (a_{\lambda,m})^2 \} N = P_0 N,\end{aligned} \tag{21}$$

where $\{a_{\lambda,j}\}$ are real coefficients in the Lebesgue spectrum expansion given in Eq. (17). Now, we can safely assume that the average value of each sequence is zero as

$$\langle \sum_{j=1}^{N} B(X_j) \rangle = 0. \tag{22}$$

This condition is automatically satisfied if the orthogonal complement $M^{\perp}$ of M corresponds to the set of constant functions as an orthonormal system constructed in Appendix. The ensemble average of l-shift auto-correlation functions is explicitly given by

$$\langle\sum_{j=1}^{N} B(X_j)B(X_{j+l})\rangle = \sum_{\lambda\in\Lambda}\sum_{m=l}^{\infty} a_{\lambda,m}a_{\lambda,m-l}. \quad (23)$$

The mean interference noise $\langle Pn\rangle$ is 0 as derived by the

$$\langle\sum_{j=1}^{N} B(X_j)B(Y_j)\rangle = N\int_{\Omega} B(x)\rho(x)dx \cdot \int_{\Omega} B(y)\rho(y)dy = 0. \quad (24)$$

Here, we assume that the initial values X_1 and Y_1 are chosen independently and they are distributed according to the invariant probability density $\rho(x)$ and $\rho(y)$. By Eq. (5) in Section 2, the mean variance of the interference noise can also be estimated as follows:

$$\begin{aligned}\langle Pn^2\rangle \equiv \langle[\sum_{j=1}^{N} B(X_j)B(Y_j)]^2\rangle &= \{\sum_{\lambda\in\Lambda}\sum_{m=0}^{\infty}(a_{\lambda,m})^2\}^2 N + \delta \\ &\geq P_0^2 N,\end{aligned}$$

where

$$\delta = 2\sum_{l=1}^{N}(N-l)(\sum_{\lambda\in\Lambda}\sum_{m=l}^{\infty} a_{\lambda,m}a_{\lambda,m-l})^2 \geq 0. \quad (25)$$

From Eq. (23), the minimum bound of the mean interference noise (interference variance) is attained when the all of the mean l-shift auto-correlation functions are zero. Such conditions which give the minimum mean interference noise are realized when

$$\langle\phi_\lambda(X_0)\phi_\lambda(X_l)\rangle = \sum_{\lambda\in\Lambda}\sum_{m=l}^{\infty} a_{\lambda,m}a_{\lambda,m-l} = 0. \quad \forall l \geq 1. \quad (26)$$

$B(x)$ satisfying the Eq. (26) are illustrated by the following examples:

$$B(x) = \phi_{\lambda,j}(x), \quad (27)$$

$$B(x) = a\phi_{\lambda_1,0}(x) + b\phi_{\lambda_1,l}(x) + c\phi_{\lambda_2,0}(x) - d\phi_{\lambda_2,l}(x), \quad (28)$$

where $ab = cd$.

Thus, not only elementary white random sequences generated by ergodic dynamical systems (27) but also the suitable sum of cchaotic sequences (28) are the optimal ones for synchronous CDMA. We assume K users have different initial conditions and correspondingly different chaotic spreading sequences. Thus, with the use of the Gaussian assumption for $K-1$ interference noises (K is large number), we finally obtain the mean SINR (signal to interference noise ratio) denoted by $R_{\text{chaos}}(K)$ as follows:

$$
\begin{aligned}
R_{\text{chaos}}(K) = \frac{\langle Ps \rangle}{\sqrt{\langle Pn^2 \rangle (K-1)}} &= \frac{P_0 N}{\sqrt{(P_0^2 N + \delta)(K-1)}} \\
&\leq \sqrt{\frac{N}{K-1}} \equiv R_{\text{chaos}^*}.
\end{aligned}
$$

On the other hand, the mean SNR in synchronous CDMA for Gold sequences of length N obtained by Tamura, Nakano, and Okazaki[8] is given by

$$
R_{\text{Gold}}(K) = \sqrt{\frac{N^3}{(K-1)(N^2+N-1)}}. \tag{29}
$$

Thus, we can say that the mean SNRs between optimal chaotic spreading sequences and Gold sequences(optimal binary sequences) have the following inequality:

$$
R_{\text{Gold}}(K) < R_{\text{chaos}^*}(K) \quad \text{for} \quad N < \infty. \tag{30}
$$

Note that the Gold sequences is *asymptotically optimal*:

$$
\lim_{N \to \infty} R_{\text{Gold}}(K)/R_{\text{chaos}^*}(K) = 1. \tag{31}
$$

A set of several periodic orbits of the chaotic sequences generated by the second order Chebyshev polynomial demonstrates this type of SNR improvement over the optimal binary sequences [9]. The present analytical result of SNR with chaotic spreading sequences can be considered as a generalization of the former analytical result in the case of Chebyshev ergodic maps [10].

6 Code Design

How many chaotic codes can we generate to use spectrum spreading sequences coexisting in a same frequency band? We consider that one bit of data is coded into one spreading sequences obtained by direct product of $s(\geq 1)$ periodic sequences of period N of chaotic dynamical systems as follows.

$$\boldsymbol{x} = \boldsymbol{y_1} \otimes \cdots \otimes \boldsymbol{y_s}. \tag{32}$$

Here, by *direct product of chaotic sequences*, we mean that each component $x(i)$ of the obtained chaotic spreading sequences $(x(1), .., x(N))$ is given by the products of s chaotic sequences:

$$x(i) = \prod_{j=1}^{s} y_j(i), \tag{33}$$

where $y_j(i)$ is the i-th $(1 \leq i \leq N)$ component of the j-th $(1 \leq j \leq s)$ chaotic spreading sequence $\boldsymbol{y_j} = (y_j(1), y_j(2), \cdots, y_j(N))$ generated by a specific chaotic dynamical system

$$y_j(i+1) = F_j(y_j(i)) \quad , j = 1, \cdots, s. \tag{34}$$

We note here that periodic sequences of such chaotic dynamical systems can also been seen as typical ergodic sequences since the ergodic equality holds in the infinite period limit with respect to the ergodic invariant measure for Axiom A dynamical systems (strongly chaotic dynamical systems) [12].

The advantage of this type of direct-product sequence construction based on multiple chaotic sequences is that we can enlarge the family size of spreading sequences so that we can enhance the communication security compared to the conventional one-dimensional binary sequences. Note while the set of s-product of ± 1 binary sequences at $s \geq 2$ is also the set of binary sequences, the set of s-product of chaotic sequences is not equal to the set of original chaotic sequences. The set of s-product chaotic sequences is not closed under the sequence product operation. Orthonormal basis functions of such s-product sequences are given by the product bases:

$$\phi_{\boldsymbol{\lambda},\, \boldsymbol{j}}(\boldsymbol{y}) = \phi_{1,\lambda_1,j}(y_1)\phi_{2,\lambda_2,j}(y_2) \cdots \phi_{s,\lambda_s,j}(y_s). \tag{35}$$

Such product basis functions in s variables are easily checked to satisfy the relation of orthonormal basis functions with Lebesgue spectrum for the s-product transformation:

$$\begin{aligned} \phi_{\boldsymbol{\lambda},\, \boldsymbol{j}\, +\, \boldsymbol{1}}(\boldsymbol{y}) &= \phi_{1,\lambda_1,j} \circ F_1(y_1) \cdots \phi_{s,\lambda_s,j} \circ F_s(y_s) \\ &= \phi_{\boldsymbol{\lambda},\, \boldsymbol{j}\, +\, \boldsymbol{1}}(F(\boldsymbol{y})). \end{aligned} \tag{36}$$

Thus, we can also give the mean SNR for this s-product sequences in terms of the coefficients of Lebesgue spectrum expansion. Each typical s-product basis function can be a sequence generator for attaining the optimal SNR as shown in the preceding section. Thus, chaotic spectrum spreading approach with this kind of chaotic code generation can add potentially many orthogonal sequences to an existing frequency band with graceful degradation. Furthermore, multi-channel CDMA with complex chaotic spreading was recently shown to be regarded as a natural extension of OFDM communications system by considering a two-dimensional chaotic dynamical system on the unit circle, which represents a complex spectrum spreading sequence with constant power [17]. In other words, CDMA and OFDM can be unified in single communications systems with complex chaotic spreading sequences. This implies that spectrum usage of CDMA and OFDM and other spectrum efficiency technologies can be managed by chaotic analysis of spectrum usage in a unified way. In this case, this implies that not only Fourier spectrum but also Lebesgue (chaotic code) spectrum, which can be a new measure of spectrum management, must be considered to attain the efficient sharing of the spectrum.

7 Chaos Approach to Market Efficiency of Spectrum Trading

The fundamental theory of economics and mathematics says that market efficiency, which can only be attained by unpredictability of trading price fluctuation with a martingale property, is the pillar hypothesis not only for conventional market but also for potential future spectrum trading. In particular, successive price movements are statistically independent fluctuation with martingale property and a sort of randomness of price fluctuation must contain as an essential ingredient for the efficient market of potential spectrum trading [8]. As is the case of financial engineering which measures the risk and expectation of stochastic variables by the Monte Carlo method, all kinds of risk measurements and expectations of chaotic fluctuating variables can be executed by chaotic Monte Carlo method using ergodic principle [4]. Furthermore, chaos can easily and systematically represent non-Gaussian fluctuations with broad probability density functions like Levy's stable law [7]. Thus, we can manage a chaotic fluctuation in spectrum trading environment, which drives the market to efficient one by the ergodic principle, and chaotic analysis and we can analytically obtain the statistical evaluation by the same ergodic principle and the chaotic analysis. Thus, it is of interest to consider the relation between chaotic spectrum spreading technology and chaotic market fluctuation. This should be related to the random property of price fluctuation in market. One can say that this kind of unpredictability of price fluctuation about spectrum sharing is easily obtained by considering the potentially infinite variety of chaotic spectrum spreading modulation methods, since randomness of price fluctuations in market can naturally considered to be a result of our lack of prediction capacity even with all information available from the past to the present time. Thus, to policy makers for spectrum management, we *recommend* that we prepare and consider potentially many spectrum sharing technologies as an option of strategy of spectrum sharing *before* putting spectrum on the market. There is a non-negligible possibility that market efficiency of the spectrum commons trading market can be obtained by a coexistence of potentially many chaotic spectrum spreading methodologies proven in the preceding sections, where statistical analysis can be performed by the above chaotic analysis while a typical price fluctuation and a combination of chaotic spectrum spreading strategies is essentially unpredictable through potential varieties of the communications methodologies sharing the spectrum.

8 Concluding Remarks

Here, we review the recent developments of employing chaos theory to communications systems towards future spectrum management. First, we provide the fundamental basis of the development, so called chaotic analysis, which expand signal in terms of complete orthonormal functions, related to Lebesgue spectrum of the underlying chaotic dynamical systems with mixing property. As an example of output of chaotic analysis, ensemble average of SNR for chaotic spreading sequences are given in terms of the expansion coefficients of Lebesgue spectrum of the corresponding ergodic transformation. For chip-asynchronous CDMA systems, the white random sequences or the complete orthogonal sequences such as Walsh-Hadamard sequences which are the optimal for chip-synchronous CDMA cannot be an optimal sequence[5]. However, as shown in [13, 15], it is possible to construct the optimal spreading sequences based on Chebyshev polynomials-type ergodic transformations with Lebesgue spectrum for chip-asynchronous CDMA. Chaotic analysis, which is a general methodology of expansion of given signals in terms of this kind of complete orthonormal functions have quite a number of applications. One of possible and important implications of chaotic analysis toward spectrum management includes an attainment of market efficiency of spectrum trading by injecting chaotic fluctuations with martingale property through potential infinite varieties of spectrum sharing methodologies with chaotic spectrum spreading sequences. Thus, we recommend that technological options or strategies for spectrum sharing must not be narrowed and potential spectrum sharing technologies should be considered as much as we can before putting spectrum on the market in order to make the spectrum market efficient. Such potentially infinitely many spectrum sharing technologies can be easily constructed by chaotic spectrum spreading technologies through recent studies by many researchers. Thus, our conclusion strongly supports multi frequency sharing technologies standards in a shared spectrum band and their coexistence of standard spectrum sharing technologies so that more operators can participate in the spectrum market and they can select their own spectrum sharing methodologies out of several standard optional technologies and the spectrum band.

A Complete Orthonormal Systems with Lebesgue Spectrum of Chebyshev Polynomials

In this appendix, we provide a particular class of complete orthonormal basis functions with Lebesgue spectrum of ergodic dynamical systems. Here, we consider a dynamical system with a Chebyshev polynomial $X_{n+1} = T_p(X_n)$ as chaotic-number generators, where $T_p(X)$ is the p-th order Chebyshev polynomial defined by $T_p[\cos(\theta)] \equiv \cos(p\theta)$ at $p \geq 2$. Examples of Chebyshev polynomials are illustrated by

$$T_1(X) = X, T_2(X) = 2X^2 - 1, T_3(X) = 4X^3 - 3X,$$
$$T_4(X) = 8X^4 - 8X^2 + 1 \cdots. \tag{37}$$

More importantly, it was shown in Ref. [3] that these Chebyshev maps T_p have mixing (thus, chaotic and ergodic) property with respect to the ergodic invariant measure $\frac{dx}{\pi\sqrt{1-x^2}}$ on the domain $\Omega = [-1, 1]$ for $p \geq 2$ and they have Lyapunov exponents $\ln p$. On the other hands, it is well-known that a system of Chebyshev polynomials constitutes a complete orthonormal system satisfying the relations

$$\int_{-1}^{1} T_i(x)T_j(x)\rho(x)dx = \delta_{i,j}\frac{(1+\delta_{i,0})}{2}, \tag{38}$$

where $\delta_{i,j}$ stands for the Kronecker delta function such that

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \tag{39}$$

Let us consider a set of functions $\{\phi_{q,j}(x)\}_{q\in\Lambda, j\geq 0}$ where

$$\phi_{q,j}(x) = \sqrt{2}T_{qp^j}(x) \tag{40}$$

and Λ is the set of positive integers $q(\geq 1)$ which satisfy

$$q \mod p \neq 0. \tag{41}$$

It is clear to see that the relation for Lebesgue spectrum

$$\phi_{q,j} \circ T_p = T_{qp^j} \circ T_p = T_{qp^{j+1}} = \phi_{q,j+1} \tag{42}$$

holds. Thus, the functional set $\{\phi_{q,j}(x)\}_{q\in\Lambda,j\geq 0}$ satisfies the definition of Lebesgue spectrum in Eq. (11). Furthermore, since the cardinality of Λ is infinite, it has *infinite* Lebesgue spectrum. It is clear to see the orthonormal relation

$$\langle \phi_{q,j}, \phi_{q',j'} \rangle = \delta_{\{(q-q')^2+(j-j')^2\},0}. \tag{43}$$

Thus, the set $\{\phi_{q,j}(x)\}_{q\in\Lambda,j\geq 0}$ (40) forms an orthonormal basis system. Since

$$\{T_{qp^j}(x)\}_{q\in\Lambda,j\geq 0} = \{T_l(x)\}_{l\geq 1}, \tag{44}$$

and

$$\int_{-1}^{1} T_{qp^j}(x) \cdot T_0(x)\rho(x)dx = 0 \quad \text{for} \quad \forall q \in \Lambda, \forall j \in \Xi, \tag{45}$$

it is shown here that the set $\{\phi_{q,j}(x)\}_{q\in\Lambda,j\geq 0}$ (40) together with the constant function $T_0(x) = 1$ can form a complete orthonormal basis of $L_2[-1,1]$. We note here the classical fact that the set of Chebyhsev polynomials can form a complete orthonormal basis of $L_2[-1,1]$.

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