Theory for Full Availability Group,

Loss System

(Exercises included)

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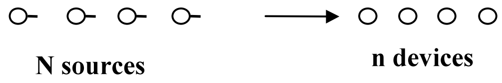
Basic Teletraffic Theory (T)

THEORY FOR FULL AVAILABILITY GROUP, LOSS SYSTEM (TFL)

Contents

1. Possible assumptions for the call intensity

Consider a full availability group with N sources and n devices in a loss system.



where N and n may be finite or infinite. Also $N \lesseqgtr n$.

The system can have p simultaneous occupations (p), where

$$0 \le p \le min\,(n, N)r \qquad \text{(TFL 1.1)}$$

In the expression

$$\left.\begin{array}{l} \lambda_p = y(p)\cdot W(p) \\ W(p) = 1 \;\; for \;\; 0 \le p < n \end{array}\right\} \qquad \text{(TFL 1.2)}$$

we have

$$W(p) = 0 \;\text{ for }\; p \ge n$$

Note that if $N \le n$, no calls are rejected since the sources cannot produce more than N simultaneous occupations.

Assumption for termination of occupations

$$\boxed{\mu_p = \frac{p}{s}} \qquad \text{(TGD 2.8)}$$

Regarding the call intensity $y(p)$, one of the following assumptions can be made:

$N \le n$	$y(p) = (N-p)\cdot\beta$	BERNOULLI (B)
$N > n$	$y(p) = (N-p)\cdot\beta$	ENGSET (EB) also called Erlang-Bernoulli
$N > n$, $N = \infty$	$y(p) = y$	ERLANG (E)
$N = \infty$, $n = \infty$	$y(p) = y$	POISSON
$n = \infty$, $n = \infty$	$y(p) = a\,(\gamma + p)$	NEG. BINOMIAL (NB)
n finite	$y(p) = a\,(\gamma + p)$	TRUNCATED NEG. BINOMIAL (TNB)

(TFL 1.3)

For the system's ability to accept a call *W(p)* at the state *(p)*, it will be assumed that

$$\left.\begin{array}{l} W(p) = 1 \text{ when } p < n, \text{ and} \\ W(p) = 0 \text{ when } p = n \end{array}\right\} \qquad \text{(TFL 1.4)}$$

meaning that all devices are occupied in the full availability group. The latter assumption $W(n) = 0$ applies only when $N \geq n$.

It will further be assumed that rejected calls will not make repeated attempts with a higher call intensity, i.e., the call intensity will not be changed by the existence of congestion. Insertion of (TFL 1.3) and (TFL 1.4) in (TGD 2.8a) and (TGD 2.9) now gives the distributions described in the sequel for the full availability group in a loss system.

2. Traffic distributions

Dependent on the choice of assumptions (TFL 1.3), different traffic distributions can now be derived:

BERNOULLI DISTRIBUTION ($N \leq n$)

Assumptions (TGD 2.8) + (TFL 1.3B) + (TFL 1.4) inserted in (TGD 2.8a) and (TGD 2.9) give

$$\boxed{\begin{array}{l} [p] = \binom{N}{p} \cdot a^p \cdot (1-a)^{N-p} \\ \\ a = \dfrac{\beta \cdot s}{1 + \beta \cdot s} \end{array}} \qquad \text{(TFL 2.1B)}$$

ENGSET DISTRIBUTION ($N > n$)

Assumptions (TGD 2.8) + (TFL 1.3EB) + (TFL 1.4) inserted in (TGD 2.8a) and (TGD 2.9) give

$$\boxed{\begin{array}{c} [p] = \dfrac{\binom{N}{p} \cdot \alpha^p}{\sum\limits_{\nu=0}^{n} \binom{N}{\nu} \cdot \alpha^\nu} \\ \\ 0 \leq p \leq n \\ \\ \alpha = \beta \cdot s \end{array}} \qquad \text{(TFL 2.1EB)}$$

ERLANG DISTRIBUTION ($N >> n$)

Assumptions (TGD 2.8) + (TFL 1.3E) + (TFL 1.4) inserted in (TGD 2.8a) and (TGD 2.9) give

$$\boxed{\begin{array}{c} [p] = \dfrac{\dfrac{A^p}{p!}}{\sum\limits_{\nu=0}^{n} \dfrac{A^\nu}{\nu!}} \\ \\ 0 \leq p \leq n \\ \\ A = y \cdot s \end{array}} \qquad \text{(TFL 2.1E)}$$

POISSON DISTRIBUTION ($N = \infty$, $n = \infty$)

Assumptions (TGD 2.8) + (TFL 1.3P) + (TFL 1.4) inserted in (TGD 2.8a) + (TGD 2.9) give

$$\boxed{\begin{gathered} [\,p\,] = \frac{A^p}{p!} \cdot e^{-A} \\ 0 \le p \le \infty \\ A = y \cdot s \end{gathered}} \qquad \text{(TFL 2.1P)}$$

NEGATIVE BINOMIAL DISTRIBUTION ($N = \infty$, $n = \infty$)

Assumptions (TGD 2.8) + (TFL 1.3NB) + (TFL 1.4) inserted in (TGD 2.8a) + (TGD 2.9) give

$$\boxed{\begin{gathered} [\,p\,] = \binom{-\gamma}{p} \cdot (-b)^p \cdot (1-b)^\gamma \\ 0 \le p \le \infty \\ b = a \cdot s \end{gathered}} \qquad \text{(TFL 2.1NB)}$$

TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION ($N = \infty$, *n finite*)

Assumptions (TGD 2.8) + (TFL 1.3TNB) + (TFL 1.4) inserted in (TGD 2.8a) + (TGD 2.9) give

$$\boxed{\begin{gathered} [\,p\,] = \frac{\binom{-\gamma}{p} \cdot (-b)^p}{\sum_{\nu=0}^{n} \binom{-\gamma}{\nu} \cdot (-b)^\nu} \\ 0 \le p \le n \\ b = a \cdot s \\ \binom{-\gamma}{\nu} = (-1)^\nu \cdot \binom{\gamma + \nu - 1}{\nu} \end{gathered}} \qquad \text{(TFL 2.1TNB)}$$

It should be noted that:

1. The Engset distribution (BE) is a truncated Bernoulli distribution (B)
2. The Erlang distribution (E) is a truncated Poisson distribution (P)

in the same way as for the two negative binomial distributions (TNB) and (NB).

It should also be noted that:

1. (B) and (EB) assume that the call intensity decreases with an increased number of occupations.
2. (P) and (E) assume a constant call intensity independent of the number of occupations.
3. (NB) and (TNB) assume that the call intensity increases with an increased number of occupations.

The most frequently used traffic distributions in the traffic theory are (B) (EB) and (E). The other three (P), (NB) and (TNB) are less used in the theory, but have been included here to illustrate how different assumptions (TFL 1.3) can provide different descriptions of the stationary traffic process. (NB) and (TNB) are sometimes used to describe the properties of overflow traffic. However, since (NB) and (TNB) are less commonly used, they will be omitted in the following text. For the derivation of expressions for the congestion, traffic carried, etc., the reader is referred to further literature on the subject.

3. Congestion

The time, E, is generally derived from the traffic distributions *[p]*, for $p = n$. Cf the definition in (TGD 3.1) and (TGD 3.3).

The call congestion, B, is derived from the traffic distributions as defined in (TGD 3.2) and (TGD 3.4).

BERNOULLI

$$E = 0 \qquad (N < m)$$

$$B = 0 \qquad (N \le m) \tag{TFL 3.1B}$$

(Since $N \le n$, no calls can be rejected)

ENGSET

$$E = [n] = \frac{\binom{N}{n} \cdot \alpha^n}{\sum_{\nu=0}^{n} \binom{N}{\nu} \cdot \alpha^\nu}$$

$$B = \frac{[n] \cdot (N-n)}{\sum_{p=0}^{n} [p] \cdot (N-p)} = \frac{\binom{N-1}{n} \cdot \alpha^n}{\sum_{\nu=0}^{n} \binom{N-1}{\nu} \cdot \alpha^\nu} \tag{TFL 3.1EB}$$

Note that $B < E$

ERLANG

$$E = \frac{\frac{A^n}{n!}}{\sum_{\nu=0}^{n} \frac{A^\nu}{\nu!}} = E_n(A) \tag{TFL 3.1E}$$

Erlang's First Formula

$B = E = E_n(A)$

($B = E$ since intensity independent of the number of occupations)

POISSON

$$\boxed{E = B = 0} \qquad \text{(TFL 3.1P)}$$

(Since $n = \infty$)

<u>*Iterative formulae*</u>

For the numerical calculation of the state probabilities *[p]*, the time congestion E and the call congestion B, as well as other characteristics of the traffic distributions, it is frequently convenient to use iterative formulae.

It already followed from (TGD 1.11) that *[p]* can be calculated from *[p-1]*, viz

$$[p] = \frac{\lambda_{p-1}}{\mu_p} \cdot [p-1]$$

if λ_{p-1} and μ_p are given their correct values. If it is not possible or practical to calculate [0], the following recursion formula may be used

$$t_p = \frac{\lambda_{p-1}}{\mu_p} \cdot t_{p-1} \qquad \text{(TFL 3.2)}$$

for $0 < p \leq m$

The recursion starts with $t_0 = 1$ and is continued to the highest possible value of p.

The state probabilities are then obtained as

$$[p] = \frac{t_p}{\sum_k t_k} \qquad \text{(TFL 3.3)}$$

which will satisfy the condition

$$[p] = 1$$

As a simple example, the recursion formula for the Erlang distribution becomes

$$\lambda_{p-1} = A$$

$$\mu_p = p$$

$$t_p = \frac{A}{p} \cdot t_{p-1}$$

$$t_0 = 1$$
$$t_1 = \frac{A}{1} \cdot 1 = A$$
$$t_2 = \frac{A}{2} \cdot t_1 = \frac{A}{2} \cdot \frac{A}{1}$$
$$t_3 = \frac{A}{3} \cdot t_2 = \frac{A}{3} \cdot \frac{A}{2} \cdot \frac{A}{1}$$
etc.

In the same way, iterative formulae can be obtained from the calculation of the congestion. Two such examples will be given here:

The Call Congestion for the Engset distribution:

$$\frac{1}{B_\nu} = 1 + \frac{1}{\alpha} \cdot \frac{\nu}{N-\nu} \cdot \frac{1}{B_{\nu-1}} \qquad \text{(TFL 3.4)}$$

where $B_0 = 1$ and the recursion is finished for $\nu = m$.

The Erlang Formula can easily be calculated from the following recursive expression

$$\frac{1}{E_\nu(A)} = 1 + \frac{\nu}{A} \cdot \frac{1}{E_{\nu-1}(A)} \qquad \text{(TFL 3.5)}$$

where $E_0(A) = 1$

Similar recursive formulae can be obtained for other expressions.

4. Traffic carried and offered

The traffic carried is defined by (TGD 3.5) and the traffic offered to the group by (TGD 3.7)

BERNOULLI

Traffic carried

$$A^1 = \sum_{p=0}^{n} p[p] = Na$$

Traffic offered

$$A = \sum_{p=0}^{n} s \cdot (N-p)\beta \cdot [p] = Na \qquad \text{(TFL 4.1B)}$$

Difference $\Delta A = A - A^1 = 0$ since $B = 0$

ENGSET

$$A^1 = \frac{N\alpha}{1+\alpha}\left(1 - \frac{N-n}{N} \cdot [n]\right) = \frac{N\alpha(1-B)}{1+\alpha(1-B)}$$

$$A = \frac{N\alpha}{1+\alpha(1-B)} \qquad \text{(TFL 4.1EB)}$$

$$\Delta A = A - A^1 = \frac{N \cdot \alpha \cdot B}{1 + \alpha \cdot (1-B)}$$

ERLANG

$$A^1 = A(1 - E_n(A))$$
$$A = N\alpha$$
$$N = \infty$$
$$\alpha = a = 0 \qquad \text{(TFL 4.1E)}$$

$$\Delta A = A - A^1 = A \cdot E_n(A)$$

POISSON

$$\boxed{\begin{array}{c} A^1 = A \\ A = N\alpha \\ N = \infty \\ \alpha = a = 0 \end{array}} \qquad \text{(TFL 4.1P)}$$

$$\Delta A = 0 \ \ (\text{since } E = B = 0)$$

Note that $A^1 = A$ when $B = 0$, i.e. when no calls are rejected from the full availability group.

5. Load on the νth device

For random hunting, every device is expected to have the same load; therefore

$$a_\nu = \frac{A^1}{n} \qquad \text{for } \nu = 1, 2, ..., n \qquad \text{(TFL 5.1)}$$

For sequential hunting, the expressions become more complicated. Only the expression for the Erlang and Poisson distributions seem to be simple, namely

$$a_\nu = A \cdot \left(E_{\nu-1}(A) - E_\nu(A)\right) \qquad \text{(TFL 5.2E)}$$

Expression (TFL 5.2E) holds also good for the Poisson distribution.

6. Improvement factor

The improvement factor which describes how much more traffic can be carried when a group is increased from n to n + Δn devices, is defined by (TGD 3.10). Expressions will be given below for an increase from n to n + 1 devices in the group.

BERNOULLI

$$F(n) = A^1(n+1) - A^1(n) = Na - Na = 0 \qquad \text{(TFL 6.1B)}$$

(Since $N \leq n$, already n devices are sufficient to carry all traffic offered to the group).

ENGSET

$$F(n,N) = \frac{N\alpha(1-B')}{1+\alpha(1-B')} - \frac{N\alpha(1-B)}{1+\alpha(1-B)}$$

where

$$B' = B(N, n+1) = \frac{\binom{N-1}{n+1} \cdot \alpha^{n+1}}{\sum_{\nu=0}^{n+1} \binom{N-1}{\nu} \cdot \alpha^\nu} \qquad \text{(TFL 6.1EB)}$$

and

$$B = B(N, n) = \frac{\binom{N-1}{n} \cdot \alpha^{n}}{\sum_{\nu=0}^{n} \binom{N-1}{\nu} \cdot \alpha^\nu}$$

ERLANG

$$F(n) = A \cdot (E_n(A) - E_{n+1}(A)) \qquad \text{(TFL 6.1E)}$$

POISSON and NEGATIVE BINOMIAL

No significance since $n = \infty$

7. Probability of X specified devices being engaged

This expression, H(x), as given in (TGD 3.11), will only be given for random hunting.

BERNOULLI

$$H(x) = \frac{\binom{N}{x}}{\binom{n}{x}} \cdot a^x \qquad \text{(TFL 7.1B)}$$

ENGSET

$$\left.\begin{aligned} H(x) &= \frac{E(n,N,\alpha)}{E(n-x,N-x,\alpha)} \\ &\textit{where} \\ E(n,N,\alpha) &= \frac{\binom{N}{n} \cdot \alpha^n}{\sum_{\nu=0}^{n} \binom{N}{\nu} \alpha^\nu} \\ E(n-x,N-x,\alpha) &= \frac{\binom{N-x}{n-x} \cdot \alpha^{n-x}}{\sum_{\nu=0}^{n-x} \binom{N-x}{\nu} \alpha^\nu} \end{aligned}\right\} \qquad (TFL\ 7.1EB)$$

ERLANG

$$H(x) = \frac{E_n(A)}{E_{n-x}(A)} \qquad \text{(TFL 7.1E)}$$

(Palm-Jacohaeus Formula)

where $E_n(A)$ is Erlang's First Formula.

POISSON

No significance since n = ∞

Note that both the expressions H(x) for EB and E have the formal appearance of conditional probabilities. The same holds also good for the truncated negative binomial distribution.

8. General comments

The previous sections have briefly presented formulae for the traffic distributions, congestion, traffic carried and offered and other typical characteristics for the traffic distributions treated here. Some formulae have been simple and others less simple. It is always an advantage if a traffic characteristic can be expressed in a simple formula. It makes it easier to understand its significance and also to make further algebraic deductions as well as to carry through numerical calculations.

The importance of simple formulae has, however, changed value with time since most traffic calculations in advanced countries are today made on electronic computers. This means that a formula that is simple for the human mind may not always be simple to the programme. Therefore, some formulae containing summation or product signs may sometimes be preferable for programming.

All distributions given above assume that the call intensity is not changed by rejected calls. This assumption gives the simplest mathematical expressions. It is, however, quite possible to introduce an assumption that traffic sources, which get calls rejected, make renewed attempts with a higher call intensity. Such an assumption would increase the realism of the traffic process description, but would also increase the mathematical complications.

However, should such an assumption be introduced, it must be kept in mind that congestion, especially in the group considered, is only a minor reason for repeated attempts. Other reasons are generally of far greater influence upon the amount of repeated call attempts.

Basic Teletraffic Theory (T)

EXERCISES A

Full availability group, lost call system. Basic exercises.

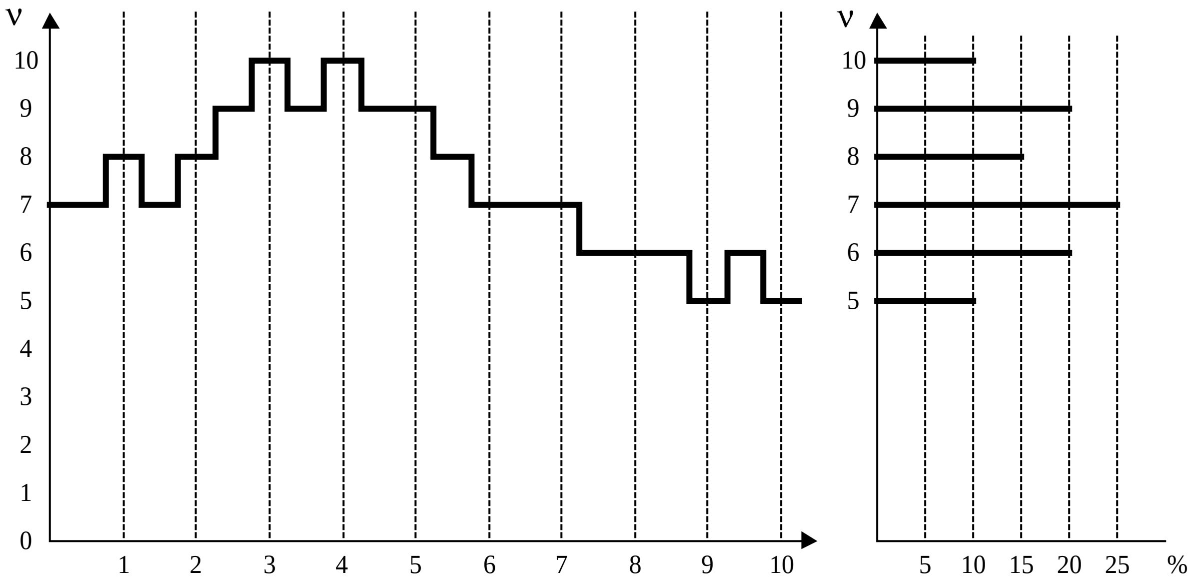
TXA 1 What is the <u>traffic offered</u>, expressed in erlangs, if the <u>calling rate</u> and the <u>mean holding time</u> are, respectively:

a) 1,000 calls/hour 90 sec.
b) 1,200 calls/hour 2 min.
c) 4 calls/sec. 1.6 min.
d) 3 calls/min. 0.04 hour

TXA 2 What is the expected number of calls per hour if the traffic offered is 35 erlangs and the mean holding time is 140 seconds?

TXA 3 What is the mean holding time in seconds if the traffic offered = 33 erlangs and the calling rate = 1,100 calls/hour?

TXA 4 In a group consisting of 10 devices, the number of busy devices was continuously read off during a period of time. The development in time is shown in the figure below. There is also given, in a diagram, the proportion of the total period when exactly 0, 1, ... , 10 devices were engaged.



a) Calculate the traffic handled by the group during the period and calculate the time congestion.

b) Suppose that the group was searched through at the points of time given in the diagram. What estimates of traffic handled and time congestion would then have been obtained?

TXA 5 Suppose that a group consists of 10 devices. Using the Erlang table, give the congestion to 4 places of decimals for the following values of A = traffic offered:

A = 1, 3, 5, 10, 15, 25, 50, 100, 200, 300 erlangs.

TXA 6 Suppose that a group is offered a traffic of 10 erlangs. Using the Erlang table, give the congestion to 4 places of decimals for the following numbers, n, of devices:

n = 1, 2, 3, 5, 7, 10, 13, 20, 23, 30.

TXA 7 What is the highest traffic in erlangs that can be offered to a group of 20 devices if the congestion is allowed to be at most 0.005?

TXA 8 How many devices should there be in a group which is offered a traffic of 48 erlangs if the congestion is allowed to be at most 0.002?

TXA 9 A group of 18 devices is offered a traffic with the calling rate 480 calls/hour ant he mean holding time 105 seconds.

What is the traffic offered, the time congestion, the call congestion, the traffic handled, the mean of the traffic handled per device and the expected number of rejected calls per hour?

TXA 10 How great a traffic is handled by the different devices in a group of 5 devices under conditions of sequential and random hunting, if the group is offered the traffic 2 erlangs?

TXA 11 Show that the mean of the traffic handled per device in a group with sequential hunting is equal to the traffic handled per device in a group with random hunting, if the groups are equally large and are offered the same traffic.

TXA 12 Determine the traffics which can be offered to groups of 10 and 100 devices if the value of congestion is to be 0.005.

Suppose that these traffics are increased by 10 % and 20 %, respectively. Calculate the resulting congestion values.

Also, give the traffic handled and the mean of the traffic handled per device for all cases.

TXA 13 Consider an Erlang loss system with n trunks, call intensity λ, and exponentially distributed conversation times with mean τ.

Let *j* be an integer $0 \le j \le n$.

a) Derive the distribution function of the time interval from an epoch when the state (j) starts until this state ends.

b) What is the mean of this time interval?

c) Calculate the probability that the state ends by a transition to $(j-1)$ and $(j+1)$ respectively.

TXA 14 A subscriber's call meter is given one pulse at the beginning of a call and then a further pulse every three minutes as long as the conversation lasts. Assume that the conversation time is an exponentially distributed variable with mean 3 minutes. Calculate the proportion of all calls that will get:

a) exactly 2 pulses

b) less than 2 pulses

c) more than 2 pulses

d) at least 2 pulses

e) at most 2 pulses

TXA 15 With the same charging principle as in Exercise TXA 14, assume a pulse interval equal to m and a mean conversation time equal to τ. Prove that the mean number of pulses per call is:

$$\frac{e^{\frac{m}{\tau}}}{e^{\frac{m}{\tau}} - 1}$$

TXA 16 On a trunk route between exchanges A and B, the mean conversation time for calls A $\rightarrow$ B is 4 minutes and for those B $\rightarrow$ A is 3 minutes. Assume for both cases exponential distribution. The calls A $\rightarrow$ B amount to 55 % of all calls. Calculate the probability that an arbitrary call will have a conversation time > 6 minutes.

TXA 17 At time 0, there are 5 conversations in progress on a trunk group. Assume that conversation times are exponentially distributed with mean 3 minutes. Calculate the probability that after 1 minute exactly 2 of those conversations are still in progress.

TXA 18 Consider an Erlang loss system with 10 trunks and offered traffic A. The trunks are hunted in order. How large a proportion of time

a) is trunk No. 9 occupied?

b) are exactly 9 trunks occupied simultaneously?

c) are the 9 first trunks occupied simultaneously?

d) are the 9 first trunks occupied simultaneously and trunk No. 10 free?

TXA 19 A full availability group consists of 6 devices and is offered traffic from 6 sources. The traffic offered per source is assumed to be 0.5 erlangs.

What is the traffic offered, the call congestion and the time congestion?

TXA 20 A full availability group consists of 6 devices and is offered traffic from a very great number of sources. The total traffic offered is assumed to be 3 erlangs. What is the call congestion and the time congestion?

TXA 21 Make a comparison between Exercises TXA 19 and TXA 20.

TXA 22

Device 1	○	●	○	●	○
Device 2	○	○	○	○	○
Device 3	○	○	●	●	●
Device 4	○	○	○	○	●
	{	123		123	
	0 engaged devices	1 engaged device		2 engaged devices	

In the above figure, some different states of occupation are shown for a group consisting of 4 devices:

a) Draw systematically all different states of occupation for this group of devices.

b) How many different states of occupation with exactly 0, 1, 2, 3, 4 engaged devices are obtained?

c) What is the total number of different states of occupation?

d) In how many of these cases are devices No. 1 and No. 3 engaged?

TXA 23 Suppose one has a group consisting of n devices and that these are numbered 1, 2, 3, ..., n.

a) How many different states are there if one is only interested in the number of engaged devices?

b) How many different states are there, if one is interested to know which of the devices are engaged?

c) How many different states of the type considered in b) are there with exactly 0, 1, 2, ..., n engaged devices?

d) How many different states of the type considered in b) are there with x specific devices engaged and, simultaneously, exactly p devices engaged?

e) Denote by [p] the probability of exactly any p devices being engaged. Use the results of c) and d) to show how H(x) shall be calculated when

$$H(x) = P\ \{x \text{ specific devices engaged}\}$$

TXA 24 Same assumptions as in Exercise TXA 19. Calculate H(3), the probability of 3 specific devices being engaged (e.g. devices Nos. 1, 2, 3 or devices Nos. 2, 4, 6).

TXA 25 Same assumptions as in Exercise TXA 20. Calculate H(3), the probability of 3 specific devices being engaged.