Statistical Model of Dangerous Insertions Dynamics in Data Package TX/RX flow

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Statement of the problem

Consider the situation when TX/RX chain has a part of "corrupted"packages. The dynamic frequencies evolution in steady state is analized.
BASIC PRESUPPOSITIONS:
- It is assumed that the (average) frequency of corrupted packages, at time instant $t + 1$ depends on average result of the present experiment at time instant t.

Рис.: TX/RX packages chain

The problem statement: How to describe the dynamics of corrupted packages frequency ?

The Model

Discrete Markov diffusion (DMD)

$$\Delta \alpha_t = -V\alpha_t + \sigma w_{t+1} \ , \quad t \geq 0, \tag{1}$$

where $\Delta\alpha_t := \alpha_{t+1} - \alpha_t$, and the stochastic component w_t, $t \geq 1$ is a stationary, in wide sense, normalized sequence of random variables witn $E[w_t] = 0$ and $E[w_t]^2 = 1$, $\forall t \geq 0$.

The aim of the communication is to answer on the following questions:

1. Stationary conditions for DMD α_t, $t \geq 1$.
2. Gaussian characterization of DMD process for the Gaussian stochastic component w_t, $t \geq 1$.
3. Statistical estimation of 2 real parameters V, σ^2 by mean of the trajectories of DMD α_t, $t \geq 1$.

Steady regime

Let $0 < V < 2$; α_0 $(E\alpha_0 = 0)$ be stationary : $E\alpha_t\alpha_{t+s} = B(s)$, uncorrelated with w_t, $t \geq 1$. Let us denote $E\alpha_0^2 = B(0) = \sigma_0^2$.

Proposition (Stationary Conditions)

The sequence α_t, $t \geq 1$ is stationary in mean square if and only if

$$\sigma_0^2 = \sigma^2/2VW \ , \quad W := 1 - V/2. \tag{2}$$

In this case

$$E\alpha_t\alpha_{t+s} = q^s\sigma_0^2 \ , \quad q := 1 - V. \tag{3}$$

Remark

Hereinafter w_t, $t \geq 1$ be i.i.d. Gaussian random variables and α_0 be a normal r.v. with $E\alpha_0 = 0$, $E\alpha_0^2 = \sigma_0^2$ and independent from w_t, $t \geq 0$.
In this case, a solution of the difference stochastic equation (1) with real-valued parameters V, σ is a discrete Gaussian Markov diffusion [Aratò 1982, DK 2014, Nevelson-Hasminsky 1973].

Covariation Statistics

We propose $(\alpha_t,\ \Delta\alpha_t)$, $t \geq 0$, as sufficient statistics allowing to estimate two parameters V, σ^2 of the DMD defined by (1).

The condition of stationarity of discrete Markov diffusion :

$$cov\,(\alpha_t, \alpha_t) = \sigma_0^2\ ,\quad \forall t \geq 0, \tag{4}$$

Proposition (2)

The cov matrix of two-component process $(\alpha_t, \Delta\alpha_t)$ *has the form:*

$$\begin{aligned} &cov\,(\alpha_t, \alpha_t) = \sigma_0^2\ ,\quad cov\,(\alpha_t, \Delta\alpha_t) = -V \cdot \sigma_0^2\ ,\quad t \geq 0, \\ &cov\,(\Delta\alpha_t, \Delta\alpha_t) = 2V \cdot \sigma_0^2\ ,\quad t \geq 1. \end{aligned} \tag{5}$$

In addition, the matrix of quadratic form for the of bivariate normal are the following:

$$\begin{aligned} &1/R_{11}^{(-1)} = W cov\,(\alpha_t, \alpha_t)\ ,\quad 1/R_{22}^{(-1)} = W cov\,(\Delta\alpha_t, \Delta\alpha_t), \\ &1/R_{12}^{(-1)} = 1/R_{21}^{(-1)} = -(2/V) W cov\,(\alpha_t, \Delta\alpha_t). \end{aligned} \tag{6}$$

A-priori statistics for DMD

The base of statistical parameters estimation, using the convergence of Gaussian stationary sequences [Bulinsky-Siryaev, 2003].

Proposition (3)

The cov. (5) generate the following statistics, unbiased and consistent (in m.s.) :

$$\begin{aligned}
\widetilde{\sigma}^2_{0T} &= \frac{1}{T}\sum_{t=1}^{T}\alpha_t^2 \xrightarrow{L^2} \sigma_0^2\ , \quad T \to \infty; \\
\widetilde{\sigma}^2_{\Delta T} &= \frac{1}{T}\sum_{t=1}^{T}(\Delta\alpha_t)^2 \xrightarrow{L^2} 2V\sigma_0^2\ , \quad T \to \infty; \\
\widetilde{\Delta}_T &= \frac{1}{T}\sum_{t=1}^{T}\alpha_t \cdot \Delta\alpha_t \xrightarrow{L^2} -V\sigma_0^2\ , \quad T \to \infty
\end{aligned} \tag{7}$$

Normal statistics for DMD

Remark

The formulas (5) give the meaning of the coefficient $V/2$:

$$V/2 = r^2 \ , \quad r = cov(\alpha_t, \Delta\alpha_t)/\sqrt{cov(\alpha_t,\alpha_t)cov(\Delta\alpha_t,\Delta\alpha_t)}. \tag{8}$$

Moreover, one has:

$$V = cov\,(\Delta\alpha_t, \Delta\alpha_t)/2\,cov\,(\alpha_t,\alpha_t)\ , \quad \sigma_0^2 = cov\,(\alpha_t,\alpha_t). \tag{9}$$

In addition, the following relation holds:

$$V = -cov\,(\alpha_t, \Delta\alpha_t)/cov\,(\alpha_t,\alpha_t). \tag{10}$$

Necessary and Sufficient conditions of convergence of Gaussian stationary sequences:

$$\frac{1}{T}\sum_{s=0}^{T-1} R^2(s) \to 0\ , \quad T \to \infty. \tag{11}$$

[Bulinsky-Siryaev, 2003].

Statistical estimators of DMD parармeters

Using the substitution method [Borovkov 1984, MS, 2:3]

Corollary

The parameters of stationarity DMD admit the following estimations:

$$\begin{aligned}
&V \approx \widetilde{V}_T = \widetilde{\sigma}^2_{\Delta T}/2\widetilde{\sigma}^2_{0T} = \sum_{t=1}^{T}(\Delta\alpha_t)^2 \Big/ 2\sum_{t=1}^{T}(\alpha_t)^2;\\
&\sigma_0^2 \approx \widetilde{\sigma}^2_{0T} = \frac{1}{T}\sum_{t=1}^{T}(\alpha_t)^2;\\
&W \approx \widetilde{W}_T = 1 - \widetilde{V}_T/2 \quad ; \quad \widetilde{C}_T = 2\cdot\widetilde{V}_T\cdot\widetilde{W}_T;\\
&\sigma^2 \approx \widetilde{\sigma}^2_T = \widetilde{C}_T\cdot\widetilde{\sigma}^2_{0T}.
\end{aligned} \tag{12}$$

Another point estimation of the parameter V coincides with max-likehood estimation

$$V \approx \widetilde{V}'_T = -\widetilde{\Delta}_T/\widetilde{\sigma}^2_{0T} = -\sum_{t=1}^{T}\alpha_t\,\Delta\alpha_t \Big/ \sum_{t=1}^{T}\alpha_t^2 \tag{13}$$

Sample likehood function of DMD

Basic relations:

$$\begin{aligned} V\alpha_t + \Delta\alpha_t &= W_\sigma(t+1)\ , \quad t \geq 0, \\ EW_\sigma(t) = 0\ , \quad & EW_\sigma^2(t) = \theta\ , \quad (\theta = \sigma^2) \end{aligned} \tag{14}$$

The logarithm of the sample likehood function:

$$L_{\theta,V}(\alpha,\,\beta) = C - \frac{1}{2}\ln\theta - \frac{1}{2\theta}\sum_{t=1}^{T}[V^2\alpha_t^2 + 2V\alpha_t\Delta\alpha_t + \Delta\alpha_t^2] \tag{15}$$

The max-likehood estimators have the following form:

$$\theta_T^* = \widetilde{\Delta\alpha_T^2} - (\widetilde{\Delta_T})^2/\widetilde{\alpha_T^2}\ , \quad V_T^* = -\widetilde{\Delta_T}/\widetilde{\alpha_T^2}, \tag{16}$$

where

$$\widetilde{\Delta\alpha_T^2} := \frac{1}{T}\sum_{t=1}^{T}(\Delta\alpha_t)^2\ , \quad \widetilde{\alpha_T^2} := \frac{1}{T}\sum_{t=1}^{T}\alpha_t^2\ , \quad \widetilde{\Delta_T} := \frac{1}{T}\sum_{t=1}^{T}\alpha_t\Delta\alpha_t. \tag{17}$$

The statistical estimators verification on simulated data

(V=0,2; σ=10)	V	σ	σ_0
Theoretical	0,2	10	16,66666667
Experimental 1	0,197199656	9,964943156	16,71275064
Experimental 2	0,202585939	10,00943022	16,5874887
Experimental 3	0,197532001	9,978812359	16,72346799
Experimental 4	0,203867079	9,944214502	16,43341025

Рис.: Statistical estimators verification on simulated data $V = 0,2;\ \sigma = 10$

(V=0,8; σ=10)	V	σ	σ_0
Theoretical	0,8	10	10,20620726
Experimental 1	0,799932079	9,979619042	10,18555019
Experimental 2	0,796208659	9,916598974	10,12916615
Experimental 3	0,802354252	9,99510257	10,19623855
Experimental 4	0,803841272	9,953497496	10,15070378

Рис.: Statistical estimators verification on simulated data $V = 0,8;\ \sigma = 10$

Kinetic fluctuation-dissipation physical model

Stokes-Einstein equation for large spherical moleculas

$$r = \frac{k_B T}{6\pi\mu D}$$

D is diffusion constant, k_B be Boltzman's constant, T be absolute temperature, μ be the medium's viscosity. r defines hydrodynamic (Stokes) radius.

$$p(x,t) = (4\pi Dt)^{-1/2}\exp(-\frac{x^2}{4Dt})$$

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Autocorrelation shape fitting method

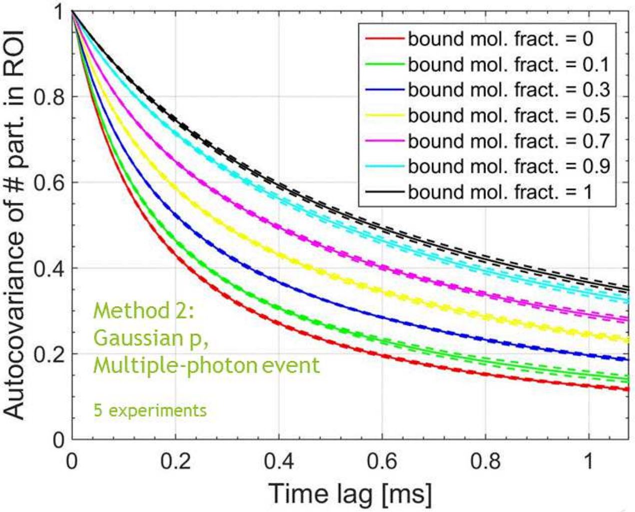

Рис.: Autocorrelation shape fitting

GSS direct application

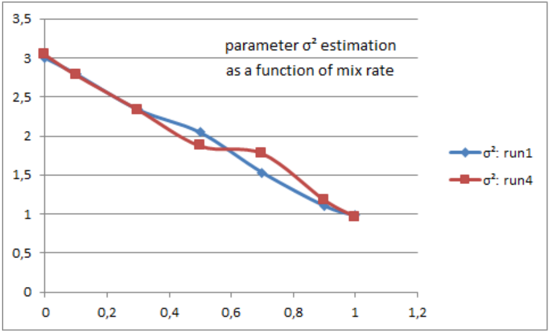

Рис.: parameter σ^2 estimated dynamics

GSS direct application

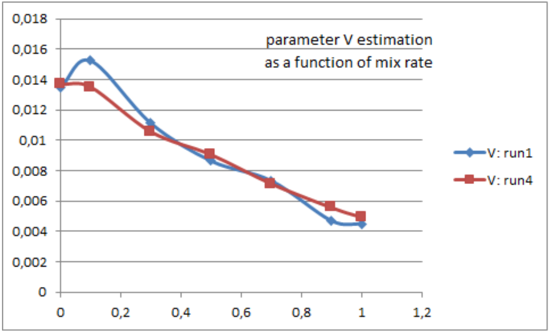

Рис.: parameter V estimated dynamics

GSS direct application

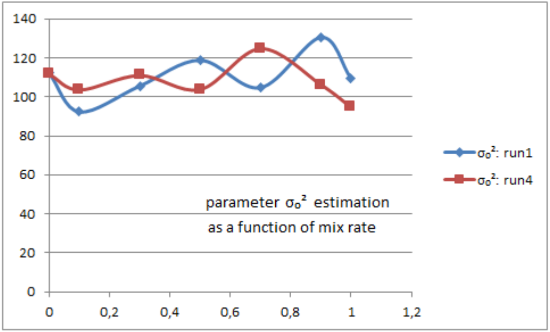

Рис.: parameter σ_0^2 estimated dynamics

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