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Radiocommunication Sector of ITU

Recommendation ITU-R TF.538-4

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Measures for random instabilities in frequency and time (phase)

TF Series

Time signals and frequency standards emissions

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Note: *This ITU-R Recommendation was approved in English under the procedure detailed in Resolution ITU-R 1.*

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RECOMMENDATION ITU-R TF.538-4

Measures for random instabilities in frequency and time (phase)

(1978-1990-1992-1994-2017)

Scope

Frequency and phase instabilities may be characterized by random processes that can be represented statistically in either the Fourier frequency domain or in the time domain. This Recommendation presents various methods and techniques for characterization of these frequency and phase instabilities.

Keywords

Random instabilities, Allan variance, time metrology, statistical measures, phase, frequency

The ITU Radiocommunication Assembly,

considering

a) that there is a need for an adequate language with which to communicate the instability characteristics of standard frequency and time sources and measurement systems;

b) that the classical variance does not converge for some of the kinds of random time and frequency instabilities;

c) that major laboratories, observatories, industries and general users have already adopted some of the Recommendations of the Technical Committee on Frequency and Time of the IEEE Society on Instrumentation and Measurement and the existence of the IEEE Standard No. 1139-2008 on "IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology – Random instabilities";

d) that frequency and time instability measures should be based on sound theoretical principles, conveniently usable, and directly interpretable;

e) that it is desirable to have frequency and time instability measures obtainable with simple instrumentation,

recommends

1 that the random instabilities of standard frequency and time signals should be characterized by the statistical measures $S_y(f)$, $S_\varphi(f)$ or $S_x(f)$ in the frequency-domain, and $\sigma_y(\tau)$, Mod. $\sigma_y(\tau)$, $\sigma_x(\tau)$, $\sigma_y(t,\tau)$, and TheoBR in the time-domain as defined below:

1.1 the measure of the normalized frequency instabilities $y(t)$ in the frequency domain is $S_y(f)$; i.e. the one-sided spectral density ($0 < f < \infty$) of the normalized frequency instabilities $y(t) = (\nu(t) - \nu_0)/\nu_0$, where $\nu(t)$ is the instantaneous carrier frequency, and ν_0 is the nominal frequency;

1.2 the measure of the phase instabilities $\varphi(t)$ in the frequency domain is $S_\varphi(f)$; i.e. the one-sided spectral density ($0 < f < \infty$) of the phase instabilities $\varphi(t)$ at a Fourier frequency f;

1.3 the measure of the phase instabilities expressed in time units (phase-time) $x(t)$ in the frequency domain is $S_x(f)$; i.e. the one-sided spectral density ($0 < f < \infty$) of phase-time instabilities $x(t)$, where $x(t) = \varphi(t) / 2\pi\, \nu_0$; $x(t)$ being related to $y(t)$ by $y(t) = \mathrm{d}x(t)/\mathrm{d}t$;

1.4 the relationships of the above spectral densities are given below:

$$S_y(f) = \frac{f^2}{\nu_0^2} S_\varphi(f) = 4\pi^2 f^2 S_x(f) \quad (1)$$

The dimensions of $S_y(f)$, $S_\varphi(f)$ and $S_x(f)$ are respectively Hz^{-1}, $Rad^2\ Hz^{-1}$ and $s^2\ Hz^{-1}$;

1.5 the measure of the normalized frequency instabilities $y(t)$ in the time domain is the two-sample standard deviation, $\sigma_y(\tau)$, and the modified two-sample standard deviation, Mod. $\sigma_y(\tau)$ and TheoBR variance as defined in Annex 1;

1.6 the measure of time instabilities in the time-domain is $\sigma_x(\tau)$ as defined in Annex 1;

1.7 the measure of the variations of the normalized frequency instabilities $y(t)$ in the time domain is the two-sample standard deviation, $\sigma_y(t,\tau)$ as defined in Annex 1;

2 that, when stating statistical measures of frequency and time instability, non-random phenomena should be recognized, e.g.:

2.1 any observed time dependency of the statistical measures should be stated;

2.2 the method of measuring systematic behaviour should be specified (e.g. an estimate of the linear frequency drift was obtained from the coefficients of a linear least squares regression to M frequency measurements, each with a specified averaging or sample time τ and bandwidth f_h);

2.3 the environmental sensitivities should be stated (e.g. the dependence of frequency and/or phase on temperature, magnetic field, barometric pressure, etc.);

3 that, when stating a measure of frequency and time instability, all relevant measurement parameters should also be specified:

3.1 the method of measurements;

3.2 the characteristics of the reference signal;

3.3 the nominal signal frequency ν_0;

3.4 the measurement system bandwidth f_h and the corresponding low pass filter response;

3.5 the total measurement time or number of measurements M;

3.6 the calculation techniques (e.g. details of lag-windows when estimating power spectral densities from time domain data, or the assumption of the effect of dead-time in estimating the two-sample standard deviation $\sigma_y(\tau)$);

3.7 the confidence of the estimate;

4 that a graphic illustration or an analytic expression of the measures of the frequency and time instabilities should be provided and should include confidence intervals when appropriate (i.e. $S_y(f)$, $S_\varphi(f)$ and $S_x(f)$ as a function of f; $\sigma_y(\tau)$, Mod. $\sigma_y(\tau)$ and $\sigma_x(\tau)$ as a function of τ; and/or $\sigma_y(t,\tau)$ as a function of t and τ).

Annex 1

Characterization of frequency and phase noise

1 Definition of terms

Frequency and phase instabilities may be characterized by random processes that can be represented statistically in either the Fourier frequency domain or in the time domain. The instantaneous, normalized frequency departure $y(t)$ from the nominal frequency ν_0 is related to the instantaneous-phase fluctuation $\varphi(t)$ about the nominal phase $2\pi \nu_0 t$ by:

$$y(t) = \frac{1}{2\pi \nu_0} \frac{d\varphi(t)}{dt} = \frac{\dot{\varphi}(t)}{2\pi \nu_0} \tag{2}$$

$$x(t) = \frac{\varphi(t)}{2\pi\nu_0}$$

where $x(t)$ is the phase variation expressed in units of time.

2 Fourier frequency domain

In the Fourier frequency domain, frequency instability may be defined by several one-sided (the Fourier frequency ranges from 0 to ∞) spectral densities such as:

$S_y(f)$ of $y(t)$, $S_\varphi(f)$ of $\varphi(t)$, $S_{\dot{\varphi}}(f)$ of $\dot{\varphi}(t)$, $S_x(f)$ of $x(t)$, etc.

These spectral densities are related by the equations:

$$S_y(f) = \frac{f^2}{\nu_0^2} S_\varphi(f) \tag{3}$$

$$S_{\dot{\varphi}}(f) = (2\pi f)^2 S_\varphi(f) \tag{4}$$

$$S_x(f) = \frac{1}{(2\pi \nu_0)^2} S_\varphi(f) \tag{5}$$

Power-law spectral densities are often employed as reasonable models of the random fluctuations in precision oscillators. In practice, it has been recognized that for many oscillators these random fluctuations are the sum of five independent noise processes and, with few limitations, the following equation is representative:

$$S_y(f) = \begin{cases} \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha & \text{for } 0 < f < f_h \\ 0 & \text{for } f > f_h \end{cases} \tag{6}$$

where h_α's are constants, α's are integers, and f_h is the high frequency cut-off of a low pass filter. Equations (3), (4) and (5) are correct and consistent for stationary noises including phase noise. High frequency divergence is eliminated by the restrictions on f in equation (6). The identification and characterization of the five noise processes are given in Table 1, and shown in Fig. 1. In practice, only two or three noise processes are usually sufficient to describe the random frequency fluctuations in a specific oscillator; the others may be neglected.

3 Time domain

Random frequency instability in the time domain may be defined by several sample variances. The square-root of a sample variance is termed a deviation, and is the statistic usually reported.

A Allan deviation $\sigma_y(\tau)$

A measure of random frequency instability is the two-sample standard deviation which is the square root of the two-sample zero dead-time variance $\sigma_y^2(\tau)$ defined as:

$$\sigma_y^2(\tau) = 1/2 \; \langle (\bar{y}_{k+1} - \bar{y}_k)^2 \rangle \tag{7}$$

where:

$$\bar{y}_k = \frac{1}{\tau} \int_{t_k}^{t_k + \tau} y(t)\, dt = \frac{x_{k+1} - x_k}{\tau} \quad \text{and} \quad t_{k+1} = t_k + \tau \quad \text{(adjacent samples)}$$

$\langle\ \rangle$ denotes an infinite time average. The measure written in equation (7) is often called the Allan variance (AVAR). The x_k and x_{k+1} are time residual measurements made at t_k and $t_{k+1} = t_k + \tau$, $k = 1, 2, ...,$ and $1/\tau$ is the fixed sampling rate which gives zero dead time between frequency measurements. By "residual" it is understood that the known systematic effects have been removed.

If the initial sampling rate is specified as $1/\tau_0$, then in general one may obtain a more efficient estimate of $\sigma_y(\tau)$ using what is called an "overlapping estimate". This estimate is obtained by computing equation (8).

$$\sigma_y^2(\tau) = \frac{1}{2(N - 2n)\,\tau^2} \sum_{i=1}^{N-2n} (x_{i+2n} - 2x_{i+n} + x_i)^2 \tag{8}$$

where N is the number of original time departure measurements spaced by τ_0 ($N = M + 1$, where M is the number of original frequency measurements of sample time, τ_0) and $\tau = n\,\tau_0$.

If dead time exists between the frequency departure measurements and this is ignored in computing equation (7), it has been shown that the resulting stability values (which are no longer the Allan variances), will be biased (except for the white frequency noise) as the frequency measurements are regrouped to estimate the stability for $n\,\tau_0$ ($n > 1$). This bias has been studied and some tables for its correction published.

If there is no dead time, then the original $\bar{y}_i$'s can be combined to create a set of $\bar{y}_k$'s:

$$\bar{y}_k = \frac{1}{n} \sum_{i=k}^{k+n-1} \bar{y}_i$$

TABLE 1

The functional characteristics of five independent noise processes for frequency instability of oscillators

Description of noise process	Slope characteristics of log-log plot				
	Frequency domain		Time domain		
	$S_y(f)$	$S_\varphi(f)$ or $S_x(f)$	$\sigma_y^2(\tau)$	Mod. $\sigma_y^2(\tau)$	$\sigma_x^2(\tau)$
	α	$\beta \equiv \alpha - 2$	μ	μ'	η
Random walk frequency	–2	–4	1	1	3
Flicker frequency	–1	–3	0	0	2
White frequency	0	–2	–1	–1	1
Flicker phase	1	–1	–2	–2	0
White phase	2	0	–2	–3	–1

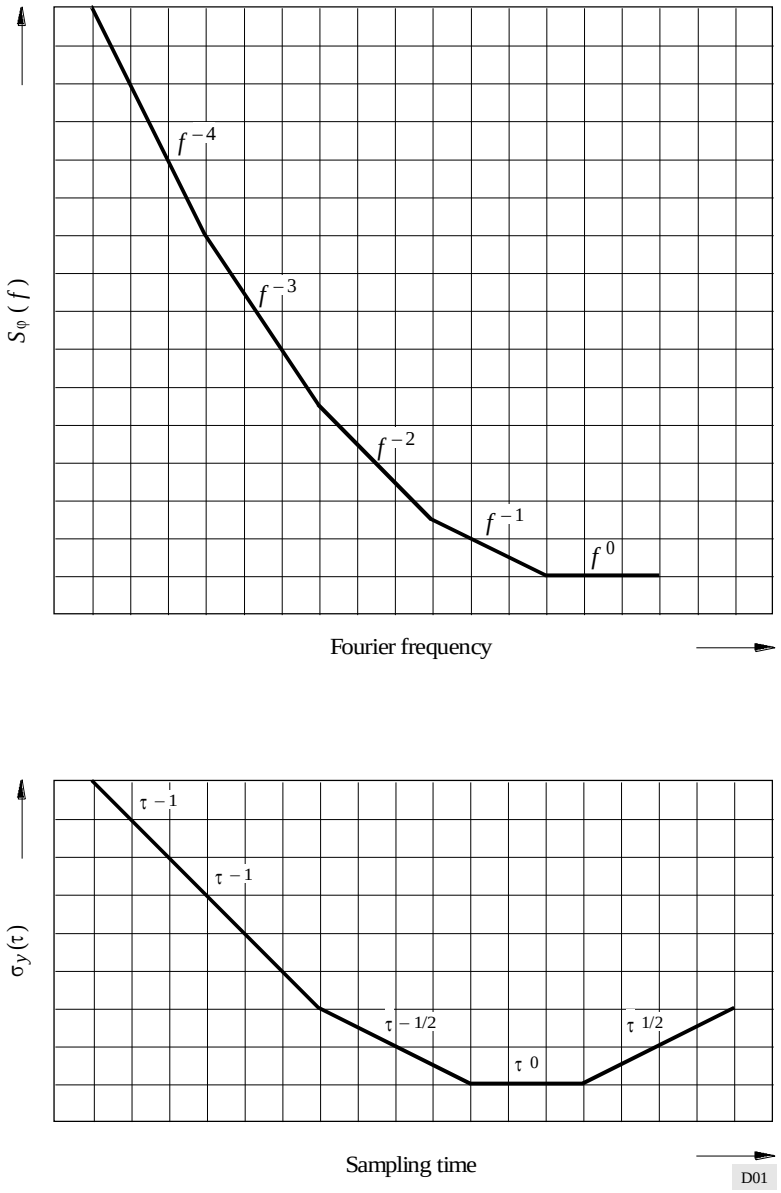
$S_y(f) = h_\alpha f^\alpha \quad \alpha = -\mu - 1, -2 \le \mu < 2 \quad \sigma_y^2(\tau) \sim |\tau|^\mu$

$S_\varphi(f) = \nu_0^2 h_\alpha f^{\alpha-2} = \nu_0^2 h_\alpha f^\beta \quad \beta \equiv \alpha - 2; \eta \equiv \mu' + 2 \quad \text{Mod. } \sigma_y^2(\tau) \sim |\tau|^{\mu'}$

$S_x(f) = \frac{1}{4\pi^2} h_\alpha f^{\alpha-2} = \frac{1}{4\pi^2} h_\alpha f^\beta \quad \alpha = -\mu' - 1 \quad \sigma_x^2(\tau) \sim |\tau|^\eta$

FIGURE 1

Slope characteristics of the five independent noise processes
(log-log scale)



An "overlapping estimate" of $\sigma_y(\tau)$ can then be obtained:

$$\sigma_y(\tau) = \left| \frac{1}{2(M - 2n + 1)} \sum_{k=1}^{M-2n+1} (\bar{y}_{k+n} - \bar{y}_k)^2 \right|^{1/2} \quad (9)$$

Thus, one can ascertain the dependence of $\sigma_y(\tau)$ as a function of τ from a single data set in a very simple way.

A plot of $\sigma_y(\tau)$ versus τ for a frequency standard typically shows a behaviour consisting of elements as shown in Fig. 1. The first part, with $\sigma_y(\tau) \sim \tau^{-1/2}$ (white frequency noise) and/or $\sigma_y(\tau) \sim \tau^{-1}$ (white or flicker phase noise) reflects the fundamental noise properties of the standard. In the case where $\sigma_y(\tau) \sim \tau^{-1}$, it is not practical to decide whether the oscillator is perturbed by white phase noise or by flicker phase noise.

Alternative techniques are suggested below. This is a limitation of the usefulness of $\sigma_y(\tau)$ when one wishes to study the nature of the existing noise sources in the oscillator. A frequency-domain analysis is typically more adequate for Fourier frequencies greater than about 1 Hz. This τ^{-1} and/or $\tau^{-1/2}$ law continues with increasing averaging time until the so-called flicker "floor" is reached, where $\sigma_y(\tau)$ is independent of the averaging time τ. This behaviour is found in almost all frequency standards; it depends on the particular frequency standard and is not fully understood in its physical basis. Examples of probable causes for the flicker "floor" are power supply voltage fluctuations, magnetic field fluctuations, changes in components of the standard, and microwave power changes. Finally the curve shows a deterioration of the stability with increasing averaging time. This occurs typically at times ranging from hours to days, depending on the particular kind of standard.

B Modified Allan deviation Mod. $\sigma_y(\tau)$

A "modified Allan variance (MVAR)", Mod. $\sigma_y^2(\tau)$, has been developed which has the property of yielding different dependences on τ for white phase noise and flicker phase noise. The dependences for "modified Allan deviation (MDEV)" Mod. $\sigma_y(\tau)$ are $\tau^{-3/2}$ and τ^{-1} respectively. Mod. $\sigma_y(\tau)$ is estimated by taking the square-root of the following equation:

$$\text{Mod. } \sigma_y^2(\tau) = \frac{1}{2\,\tau^2\,n^2\,(N - 3n + 1)} \sum_{j=1}^{N-3n+1} \left[\sum_{i=j}^{n+j-1} (x_{i+2n} - 2x_{i+n} + x_i) \right]^2 \qquad (10)$$

where:

N: original number of time variation measurements spaced by τ_0

$\tau = n\,\tau_0$ the sample time of choice.

Properties and confidence of the estimate are discussed in the technical literature. Maximum likelihood methods of estimating $\sigma_y(\tau)$ for the specific models of white frequency noise and random walk frequency noise have been developed. These two models have been shown to be useful for sample times longer than a few seconds for caesium beam standards.

C Time deviation $\sigma_x(\tau)$

The time instability in the time-domain for the five independent noise processes shown in Fig. 1 may be measured using the second-difference of adjacent time averages. This measure is also related to Mod. $\sigma_y^2(\tau)$.

$$\sigma_x^2(\tau) = (\tau^2 \text{ Mod. } \sigma_y^2(\tau)) / 3 \qquad (11)$$

$$\sigma_x^2(\tau) = (1/6) < [\bar{x}_{k+2n} - 2\,\bar{x}_{k+n} + \bar{x}_k]^2 > \qquad (12)$$

where $(dx/dt) = y$ and $\tau = n\,\tau_0$. Therefore, $\sigma_x(\tau)$ is the time deviation (TDEV). The brackets "< >" denote an infinite time average, and the bar "–" over the x denotes an average over an interval τ. The three averages used in the second difference equation above are adjacent. Therefore, for a given value of k in the second difference above, these averages occupy a space of 3τ.

The spectral density and time variance relationships are as follows:

$$\begin{aligned} S_x(f) &\sim f^{\beta} \\ \sigma_x^2(\tau) &\sim \tau^{\eta} \\ \beta &= -\eta - 1 \end{aligned} \qquad (13)$$

Since the usual types of measurement noise are centred around $\eta = 0$, TVAR gives a near-zero dependence on τ (a desirable trait for a good measure). Other useful characteristics of this measure are:

– it is equal to the classical standard deviation of the time difference measurements for $\tau = \tau_0$, for white phase modulation (WPM);

– it equals the standard deviation of the mean of the time difference measurements for $\tau = N\,\tau_0$ (the data length), for WPM;

– it is convergent and well behaved for the random processes commonly encountered in time and frequency metrology;

– the τ dependence indicates the power-law spectral-density model appropriate for the data;

– the amplitude of $\sigma_x(\tau)$ at a particular value of τ, along with assumption of one of the five power-law spectral-density models ($\beta = -4, -3, -2, -1, 0$), provides enough information to estimate the corresponding level in the frequency domain for any of the standard spectral-density measures.

The problem of estimating the stability of individual clocks from comparison measurements has been studied proposing a general and consistent model to deal with signal difference measurements, without any *a priori* assumption of lack of correlation between clocks.

D Dynamic Allan deviation $\sigma_y(t,\tau)$

The dynamic Allan deviation (DADEV) is a measure of the variations of the random frequency instability in the time domain, and it is the square root of the dynamic Allan variance (DAVAR), defined as:

$$\sigma_y^2(t,\tau) = \frac{1}{2(T_w - 2\tau)} \int_{t-\frac{T_w}{2}+\tau}^{t+\frac{T_w}{2}-\tau} \mathcal{E}\left[\left(\bar{y}(t'+\tau) - \bar{y}(t')\right)^2\right] dt' \tag{14}$$

where $\mathcal{E}$ denotes the expected value obtained as an ensemble average over an infinite number of experimental realizations, and T_w is the length of the analysis window. An estimate of the DAVAR can be obtained by:

$$\sigma_y^2(n,k) = \frac{1}{2k^2\tau_0^2(N_w - 2k)} \sum_{m=n-\frac{N_w}{2}}^{m=n+\frac{N_w}{2}-2k-1} (x_{m+2k} - 2x_{m+k} + x_m)^2 \tag{15}$$

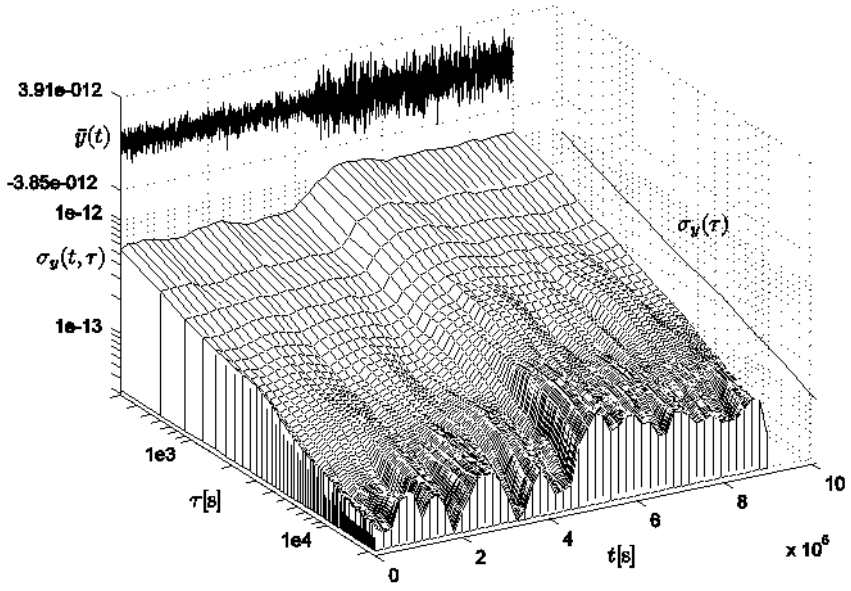
where $t = n\tau_0$, $\tau = k\tau_0$, $T_w = N_w\tau_0$, and N_w is assumed to be even.

Experimental evidence shows that the instability of clocks and oscillators, whose standard measure is the Allan variance, can change with time due to several factors, such as temperature, humidity, vibrations, gravitational effects, and radiations. The dynamic Allan variance (DAVAR) measures such variations with time of the clock instability.

Figure 2 shows the average frequency deviation $\bar{y}(t)$ of a white frequency noise whose variance increases suddenly. The DAVAR, whose square root $\sigma_y(t,\tau)$ is shown in the mesh plot, reveals the change of variance in the noise, whereas the Allan variance, whose square root $\sigma_y(\tau)$ is shown in the side plot, averages out the change of variance.

FIGURE 2

DAVAR of a change of variance (mesh plot)



E A long-term estimator TheoBR

A long-term estimator for the Allan variance (AVAR) is the Theory variance #1, known as Theo1, and with bias removed, known as TheoBR variance, which is Theo1 with a slight bias removed relative to sample AVAR. Theo1 averages every permissible squared second-difference of time errors x_i in a given data run N_x and reports frequency stability 50% longer than the longest possible τ of AVAR. Thus, Theo1 and TheoBR represent a new species of descriptive statistics with substantially higher equivalent degrees of freedom than the overlapping version of sample AVAR.

Theo1 Definition

$$Theo1(m,\tau_0,N_x) \quad := \frac{1}{0.75(N_x-m)(m\tau_0)^2} \times \sum_{i=1}^{N_x-m} \sum_{\delta=0}^{\frac{m}{2}-1} \frac{1}{\left(\frac{m}{2}-\delta\right)} \left[\left(x_i - x_{i-\delta+\frac{m}{2}}\right) + \left(x_{i+m} - x_{i+\delta+\frac{m}{2}}\right)\right]^2 \qquad (16)$$

$$for\ m\ even, 10 \le m \le N_x, \tau = 0.75m\tau_0$$

Theo1 deviation, known as Theo1-dev, is unbiased relative to the Allan deviation for White-FM noise. Theo1-dev is biased slightly for other noise types. This bias can be automatically removed using an algorithm that computes the average bias between Theo1 and Allan variance in an overlapping τ region of the two and uses it to correct Theo1. Thus, we have:

$$TheoBR = \frac{\mathcal{E}[*AVAR]}{\mathcal{E}[*Theo1]} \times Theo1 \qquad (17)$$

where *AVAR and *Theo1 are computed for a range of $m\tau_0$ where sample AVAR and Theo1 overlap and are chosen so as to represent the bias at large τ for AVAR rather than at small τ. The bias correction is a constant, not dependent on τ.

TheoBR Definition

$$TheoBR(m,\tau_0,N_x) \ := \left[\frac{1}{n+1}\sum_{i=1}^{n} \frac{AVAR(m=9+3i,\tau_0,N_x)}{Theo1(m=12+4i,\tau_0,N_x)}\right] \times Theo1(m,\tau_0,N_x) \qquad (18)$$

$$where\ n = \left\lfloor \frac{0.5N_x}{3} - 3 \right\rfloor$$

TheoH Definition

Hybrid Theory variance, known as TheoH variance, is a hybrid statistic (hence, the "H") that combines values of the overlapping version of AVAR in the short-term and TheoBR variance to estimate the long-term frequency stability out to ¾ of the data run length N_x, which as mentioned is 50% longer than the data length usable by AVAR.

$$TheoH(m,\tau_0,N_x) := \begin{cases} AVAR(m,\tau_0,N_x), & for\ 1 \le m \le \frac{k}{\tau_0} \\ TheoBR(m,\tau_0,N_x), & for\ \frac{k}{0.75\tau_0} \le m \le N_x - 1, m\ even \end{cases} \quad (19)$$

where k is $\tau \le 0.2T$, and $AVAR(m,\tau_0,N_x)$ has sufficient confidence.

The usual upper τ-limit of AVAR is 20% of the data run time *T*. Since TheoBR is more computationally intense than AVAR, only large values of *m* are included in the above definition, not all values of *m*. Usually it is not necessary to calculate all values of *m*; octave or decade spans of *m* are sufficient to characterize random noise. To delineate AVAR from TheoBR values, an intentional small gap for $\frac{0.2T}{\tau_0} < m < \frac{0.8T}{3\tau_0}$ is inserted but can be filled if one wants to compute more or all *m* values.

The square-root of TheoH variance is TheoH deviation, also known as TheoH-dev.

4 Conversion between frequency and time domains

In general, if the spectral density of the normalized frequency fluctuations $S_y(f)$ is known, the two-sample variance can be computed as follows:

$$\sigma_y^2(\tau) = 2 \int_0^{f_h} S_y(f)\ \frac{\sin^4(\pi f \tau)}{(\pi f \tau)^2}\ df \quad (20)$$

$$\text{Mod. } \sigma_y^2(\tau) = 2 \int_0^{f_h} S_y(f)\ \frac{\sin^6(\pi f \tau)}{(n\,\pi f \tau)^2 \sin^2(\pi f \tau_0)}\ df \quad (21)$$

and:

$$\sigma_x^2(\tau) = \frac{8}{3} \int_0^{f_h} S_x(f)\ \frac{\sin^6(\pi f \tau)}{n^2 \sin^2(\pi f \tau_0)}\ df \quad (22)$$

Specifically, for the power law model given by equation (6), the time-domain measure also follows the power law as derived from equations (6) and (11).

$$\sigma_y^2(\tau) = h_{-2}\ \frac{(2\pi)^2}{6}\ \tau + h_{-1}\ 2 \log_e 2 + h_0\ \frac{1}{2\tau} + h_1\ \frac{1.038 + 3 \log_e(2\pi f_h \tau)}{(2\pi)^2\ \tau^2} + h_2\ \frac{3 f_h}{(2\pi)^2\ \tau^2} \quad (23)$$

The values of h_α are characteristics of oscillator frequency noise. One may note for integer values (as often seems to be the case) that

$$\mu = -\alpha - 1 \qquad \text{for } -3 \le \alpha \le 1$$

$$\mu \sim -2 \qquad \text{for } \alpha \ge 1$$

where:

$$\sigma_y^2(\tau) \sim \tau^{\mu}$$

These conversions have been verified experimentally and by computation. Table 2 gives the coefficients of the translation among the frequency instability measures from time domain to frequency domain and from frequency domain to time domain.

TABLE 2

Translation of frequency instability measures from spectral densities in frequency domain to variance in time domain and vice versa (for $2\pi f_h \tau \gg 1$)

Description of noise processes	$\sigma_y^2(\tau) =$	$S_y(f) =$	$S_\varphi(f) =$	$S_x(f) =$
Random walk frequency	$A[f^2 S_y(f)]\,\tau^1$	$\frac{1}{A}[\tau^{-1}\sigma_y^2(\tau)]\,f^{-2}$	$\frac{\nu_0^2}{A}[\tau^{-1}\sigma_y^2(\tau)]\,f^{-4}$	$\frac{3.636}{A}[\tau^{-3}\sigma_x^2(\tau)]\,f^{-4}$
Flicker frequency	$B[f S_y(f)]\,\tau^0$	$\frac{1}{B}[\tau^2\sigma_y^2(\tau)]\,f^{-1}$	$\frac{\nu_0^2}{B}[\tau^0\sigma_y^2(\tau)]\,f^{-3}$	$\frac{0.741}{AB}[\tau^{-2}\sigma_x^2(\tau)]\,f^{-3}$
White frequency	$C[f^0 S_y(f)]\,\tau^{-1}$	$\frac{1}{C}[\tau^2\sigma_y^2(\tau)]\,f^0$	$\frac{\nu_0^2}{C}[\tau^1\sigma_y^2(\tau)]\,f^{-2}$	$\frac{1}{AC}[\tau^{-1}\sigma_x^2(\tau)]\,f^{-2}$
Flicker phase	$D[f^{-1} S_y(f)]\,\tau^{-2}$	$\frac{1}{D}[\tau^2\sigma_y^2(\tau)]\,f^1$	$\frac{\nu_0^2}{D}[\tau^2\sigma_y^2(\tau)]\,f^{-1}$	$0.89\,[\tau^0\sigma_x^2(\tau)]\,f^{-1}$
White phase	$E[f^{-2} S_y(f)]\,\tau^{-2}$	$\frac{1}{E}[\tau^2\sigma_y^2(\tau)]\,f^2$	$\frac{\nu_0^2}{E}[\tau^2\sigma_y^2(\tau)]\,f^0$	$\frac{1}{\tau_0 f_h}[\tau\sigma_x^2(\tau)]\,f^0$

$A = \frac{4\pi^2}{6}$ $\quad$ $D = \frac{1.038 + 3\log_e(2\pi f_h \tau)}{4\pi^2}$

$B = 2\log_e 2$ $\quad$ $E = \frac{3 f_h}{4\pi^2}$

$C = 1/2$

The slope characteristics of the five independent noise processes are plotted in the frequency and time domains in Fig. 1 (log-log scale).

5 Confidence limits of time domain measurements

To estimate the confidence interval or error bar for a Gaussian type of noise of a particular value $\sigma_y(\tau)$ obtained from a finite number of samples, the following expression may be used (with non-overlapping estimates):

$$\text{Confidence Interval } I_\alpha \simeq \sigma_y(\tau) \cdot \kappa_\alpha \cdot M^{-1/2} \quad \text{for} \quad M > 10 \tag{24}$$

where:

M: total number of data points used in the estimate

α: as defined in the previous section

$\kappa_2 = \kappa_1 = 0.99$

$\kappa_0 = 0.87$

$\kappa_{-1} = 0.77$

$$\kappa_{-2} = \quad 0.75.$$

As an example of the Gaussian model with $M = 100$, $\alpha = -1$ (flicker frequency noise) and $\sigma_y(\tau = 1\ \text{s}) = 10^{-12}$, one may write:

$$I_\alpha \simeq \sigma_y(\tau) \cdot \kappa_\alpha \cdot M^{-1/2} = \sigma_y(\tau) \cdot (0.77) \cdot (100)^{-1/2} = \sigma_y(\tau) \cdot (0.077) \tag{25}$$

which gives:

$$\sigma_y(\tau = 1\ \text{s}) = (1 \pm 0.08) \times 10^{-12} \tag{26}$$

A modified estimation procedure including dead time between pairs of measurements has also been developed, showing the influence of auto-correlation of frequency fluctuations.

The above confidence intervals apply to "non-overlapping estimates". In the case of "overlapping estimates" the confidence interval is smaller and can be calculated.

The bias resulting from the application of two-sample variance to time intervals obtained by linking several successive measures with dead time has been determined as a function of noise type. This bias may be significant.

The effect of the nature of the analogue filtering which limits the noise power of the signal in question about its nominal frequency has been determined, particularly for the use of a low pass filter instead of a band-pass filter centred on the nominal frequency.

The degrees of freedom (d.f.) for overlapping estimates have been calculated. They are theoretically derived and plotted for power-law spectra for estimation of the confidence interval of the two-sample standard deviation. The confidence interval for the two-sample standard deviation $\sigma_y(\tau)$ is:

$$\frac{\sqrt{(d.f.)\ \hat{\sigma}_y^2(\tau)}}{\chi_{P_1}} < \sigma_y(\tau) < \frac{\sqrt{(d.f.)\ \hat{\sigma}_y^2(\tau)}}{\chi_{P_2}} \tag{27}$$

where:

χ_{P_1} and χ_{P_2} : percentile values for the chi-square distribution;

the hat "^" : estimate or the measured two-sample variance from finite set.

For $\alpha = +2$ the improvement of the d.f. is nearly n times better than with respect to the non-overlapping estimate case. Significant improvement is also gained for $\alpha = +1$. For $\alpha = 0$ the ratio of the degrees of freedom is 2; for $\alpha = -1$ it is 1.3; and for $\alpha = -2$ it is 1.04.

6 Potentially misleading applications of statistical measures

The several statistical measures specified here are each created for specific purposes, and they may not be useful for others.

The Allan deviation, modified Allan deviation, and time deviation are based upon second differences of $x(t)$ and are therefore insensitive to a first-order polynomial. However, removal of a polynomial of second or higher order can reduce the values at large τ ($\tau > 0.2T$).

Therefore, if such a polynomial must be removed before computing the statistics, it is strongly recommended to use statistical measures insensitive to the curve removal, such as those based upon higher order differences of $x(t)$. The class of statistical measures that use second or higher-order differences are insensitive to a linear variation in the data.

Both the frequency-domain and the time-domain measures may give incorrect results in the presence of irregularly spaced data. In general, gaps in the data constitute missing information. Some methods for dealing with these data use interpolation, others assume missing data are zero, and all can cause distorting effects. The ways in which software handles irregularly spaced data should be understood.

7 Conclusion

The statistical methods for describing frequency and phase instability and the corresponding power law spectral density model described are generally sufficient for describing oscillator instability.. The current version introduces additional methods to deal with time varying instabilities in the time domain, and extends the computation of time domain instability to a larger fraction of the data length.

Non-random (deterministic or systematic) variations are not covered by the methods described. These could be either periodic or monotonic. Periodic variations are to be analysed by means of known methods of harmonic analysis. Monotonic variations are described by linear or higher order drift terms.