International Telecommunication Union

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Radiocommunication Sector of ITU

Recommendation ITU-R TF.2118-0
(12/2018)

Relativistic time transfer

TF Series
Time signals and frequency standards emissions

International Telecommunication Union

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Note: *This ITU-R Recommendation was approved in English under the procedure detailed in Resolution ITU-R 1.*

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RECOMMENDATION ITU-R TF.2118-0

Relativistic time transfer

(2018)

Scope

This Recommendation establishes common conventional algorithms and procedures to be used in comparing clocks on the surface of the Earth and on platforms far from the Earth but within the solar system. These expressions are explicitly determined in the general relativity theory that is presently accepted to form the basis of space-time reference systems. It is envisioned that these algorithms and procedures would be used for comparisons of clocks on Earth satellites, interplanetary spacecraft, and on the surfaces of solar system bodies.

Keywords

Relativity, time transfer, signal propagation, clock, coordinate system

Related Recommendations and Reports

Recommendations ITU-R TF.1011-1, ITU-R TF.767-2, ITU-R TF.374-5

The ITU Radiocommunication Assembly,

considering

a) that it is desirable to maintain coordination of standard time and frequency on platforms operating in the vicinity of the Earth and in the solar system;

b) that accurate means of transferring time and frequency are required to meet the future needs of communication, navigation, and science in the vicinity of the Earth and in the solar system;

c) that clocks are subject to path dependent time and frequency variations due to their motion and to the gravitational potential in which they operate;

d) that the conceptual foundation for the transfer of time and frequency should be clearly outlined;

e) that procedures for the transfer of time and frequency in the vicinity of the Earth and across celestial bodies and spacecraft in the solar system require the use of mathematical algorithms that account for relativistic effects,

f) that requirements for precision and accuracy for the transfer of time and frequency in the vicinity of the Earth and in the solar system depend on the specific application,

recommends

that the mathematical algorithms that account for relativistic effects in the transfer of time and frequency as provided in Annex I be used as appropriate.

Annex I

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1 Objective

The objective of Annex I is to outline the basic concepts and procedures to be applied to address the effects of relativity in timekeeping, navigation, science, and communication systems. It is intended that this may serve as a convenient reference for specific applications, including the comparison of times registered by clocks on spacecraft in orbit around the Earth, in interplanetary space, and on planetary surfaces with the times recorded by clocks on Earth. It is based on the IERS Conventions (2010), the ITU-R Handbook on Satellite Time and Frequency Transfer and Dissemination (2010), Nelson, Metrologia (2011), and Petit and Wolf, Metrologia (2005). Users may consult those publications and references cited therein for further details.

2 Relativistic Framework

A reference system is a mathematical construction used to specify space-time events described by four coordinates $x^\alpha = (x^0, x^i) = (x^0, x^1, x^2, x^3)$. The Greek indices take the values 0, 1, 2, 3 and the Latin ones take the values 1, 2, 3, and a repeated index implies summation on that index. The index 0 refers to the time variable and the indices 1, 2, 3 refer to the three spatial coordinates. A reference frame is a realization of the reference system usually in the form of a catalogue of positions and motions of objects or an ephemeris.

A reference system is fixed by its metric tensor, $g_{\alpha\beta}(t, x^i)$ allowing one to compute the space-time distance between two events x^α and $x^\alpha+dx^\alpha$:

$$ds^2 = g_{\alpha\beta}\left(t, x^i\right) dx^\alpha dx^\beta \equiv g_{00}\mathrm{c}^2 dt^2 + 2g_{0i}\mathrm{c}dtdx^i + g_{ij}dx^i dx^j . \qquad (1)$$

where c is the speed of light.

Resolutions of international scientific organizations define the reference system specifically. The most important are:

1) IAU Resolution A4 (1991) defines the Barycentric Celestial Reference System (BCRS) and the Geocentric Celestial Reference System (GCRS) and their time coordinates. IAU Resolution B1 (2000) further refines the BCRS definition.

2) IUGG Resolution 2 (2007) defines the Geocentric Terrestrial Reference System (GTRS), along with the International Terrestrial Reference System (ITRS).

Briefly, the BCRS is a system of space-time coordinates for the solar system centred at the solar system barycentre and specified with the metric tensor given by the IAU 2000 Resolution B1.3. (See https://www.iau.org/administration/resolutions/general_assemblies/.) The GCRS is an Earth-centred, inertial (ECI) system of geocentric space-time coordinates centred at the geocenter with metric tensor also specified by the IAU 2000 Resolution B1.3. It is defined such that the transformation between BCRS and GCRS spatial coordinates contains no rotation component, so that GCRS is kinematically non-rotating with respect to BCRS. The Geocentric Terrestrial Reference System (GTRS) is an Earth-Centred Earth-Fixed (ECEF) coordinate system.

In the framework of general relativity, proper time (τ) refers to the actual reading of a clock or the local time in the clock's own frame of reference. Coordinate time (t) refers to the independent variable in the equations of motion of material bodies and in the equations of propagation of electromagnetic waves. It is a mathematical coordinate in the four-dimensional space-time of the coordinate system. For a given event, the coordinate time has the same value everywhere. Coordinate times are not measured; rather, they are computed from the proper times of clocks. The relation between coordinate time and proper time depends on the clock's position and state of motion in its gravitational environment and is derived by integration of the space-time interval. In the comparison of the proper times of two clocks, the coordinate time is ultimately eliminated. Thus the relativistic transfer of time between clocks is independent of the coordinate system. The coordinate system may be chosen arbitrarily on the basis of convenience.

For a transported clock, the space-time interval is:

$$ds^2 = g_{00}\,\mathrm{c}^2\,dt^2 + 2\,g_{0j}\,\mathrm{c}\;dt\,dx^j + g_{ij}\,dx^i\,dx^j = -\mathrm{c}^2 d\tau^2 \qquad (2)$$

Thus $dt = d\tau$ for a clock at rest in an inertial frame of reference, for which $dx^i = 0$ and $-g_{00} = 1$, $g_{0j} = 0$, and $g_{ij} = \delta_{ij}$. The elapsed coordinate time corresponding to the measured proper time during the transport of a clock between points A and B is:

$$\Delta t = \pm\int_A^B \frac{1}{\sqrt{-g_{00}}}\sqrt{1+\frac{1}{\mathrm{c}^2}\left(g_{ij}+\frac{g_{0i}\,g_{0j}}{-g_{00}}\right)\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}\;d\tau \;+\; \frac{1}{\mathrm{c}}\int_A^B \frac{g_{0j}}{-g_{00}}\,\frac{dx^j}{d\tau}\,d\tau\,. \qquad (3)$$

For an electromagnetic signal, the space-time interval is:

$$ds^2 = g_{00}\,\mathrm{c}^2\,dt^2 + 2\,g_{0j}\,\mathrm{c}\;dt\,dx^j + g_{ij}\,dx^i\,dx^j = 0 \qquad (4)$$

The speed of light is c in every inertial frame of reference. The elapsed coordinate time of propagation between points A and B is

$$\Delta t = \pm\frac{1}{\mathrm{c}}\int_A^B \frac{1}{\sqrt{-g_{00}}}\sqrt{\left(g_{ij}+\frac{g_{0i}\,g_{0j}}{-g_{00}}\right)dx^i\,dx^j} \;+\; \frac{1}{\mathrm{c}}\int_A^B \frac{g_{0j}}{-g_{00}}\,dx^j \qquad (5)$$

3 Time-scales

Coordinate Time-scales

Geocentric Coordinate Time (TCG) is the coordinate time in a coordinate system with origin at the Earth's centre.

Terrestrial Time (TT) is a coordinate time in the ECI coordinate system that is rescaled from TCG so that it has approximately the same rate as the proper time of a clock at rest on the geoid. The relationship between TCG and TT is defined such that $d\mathrm{TT}/d\mathrm{TCG} \equiv 1 - L_G$, where L_G is a defined constant $\equiv 6.969\,290\,134 \times 10^{-10} \approx 60.2$ µs/d. Consequently:

$$\mathrm{TT} = (1 - L_G)\,\mathrm{TCG} \qquad (6)$$

or

$$\mathrm{TCG} - \mathrm{TT} \;=\; L_G\,\mathrm{TCG} \;=\; \frac{L_G}{1-L_G}\,\mathrm{TT} \qquad (7)$$

Barycentric Coordinate Time (TCB) is the coordinate time in a coordinate system with origin at the solar system barycentre. The difference between TCB and TCG is of the form:

$$\mathrm{TCB} - \mathrm{TCG} \;=\; L_C\,\mathrm{TCB} + P(t) + \frac{1}{\mathrm{c}^2}\,\mathbf{v}_E(t)\cdot\mathbf{R}(t) \qquad (8)$$

where $L_C = 1 - \langle d\mathrm{TCG}/d\mathrm{TCB}\rangle$, $\langle d\mathrm{TCG}/d\mathrm{TCB}\rangle$ being an average offset in rate, $P(t)$ represents a series of periodic terms, $\mathbf{v}_E(t)$ is the barycentric velocity of the Earth's centre of mass, and $\mathbf{R}(t)$ is the time dependent position vector with respect to the geocenter.

The net transformation from TCB to TT has an average offset in rate:

$$\langle d\mathrm{TT}/d\mathrm{TCB}\rangle = (d\mathrm{TT}/d\mathrm{TCG})\langle d\mathrm{TCG}/d\mathrm{TCB}\rangle = (1 - L_G)(1 - L_C) \equiv 1 - L_B \qquad (9)$$

where $L_G \equiv 6.969\ 290\ 134 \times 10^{-10} \approx 60.2$ µs/d, $L_C = 1.480\ 826\ 867\ 41 \times 10^{-8} \approx 1.28$ ms/d, and $L_B = 1.550\ 519\ 767\ 72 \times 10^{-8} \approx 1.34$ ms/d. The difference between TCB and TT is:

$$\mathrm{TCB} - \mathrm{TT} = L_B\,\mathrm{TCB} + (1 - L_G)\left[P(t) + \frac{1}{c^2}\mathbf{v}_E(t)\cdot\mathbf{R}(t)\right] \tag{10}$$

The epoch of TT, TCG, and TCB is January 1, 1977 0 h 32.184 s TAI (JD 2 443 144.5003725).

Barycentric Dynamical Time (TDB) is a time-scale rescaled from TCB, defined by the expression $\mathrm{TDB} \equiv (1 - L_B)\,\mathrm{TCB} + \mathrm{TDB}_0$, where $L_B \equiv 1.550\ 519\ 768 \times 10^{-8}$ and $\mathrm{TDB}_0 \equiv -65.5$ µs are defining constants. TDB has the same rate as TT.

Atomic Time-scales

International Atomic Time (TAI) is the fundamental scale of time based on atomic clocks which is calculated at the Bureau International des Poids et Mesures (BIPM) from a weighted average of the readings of atomic clocks in timing laboratories distributed around the world. The process comprises two steps: (1) a free atomic time-scale, Echelle Atomique Libre (EAL), is calculated using the clock comparison data; (2) frequency corrections are applied to EAL based on the readings of primary frequency standards operated in a few laboratories, reduced by relativistic corrections to the conventionally defined geoid. TAI is a continuous reference time-scale that is not disseminated. Although TT is a theoretical uniform time-scale and TAI is derived statistically, a practical realization for TT is TT = TAI + 32.184 s.

Coordinated Universal Time (UTC) is the atomic time-scale for civil timekeeping which differs from TAI by an integral number of leap seconds. UTC is disseminated every month in BIPM *Circular T* in the form of the differences from individual laboratory realizations UTC(k), where k is the designation of the laboratory involved.

4 Clock Comparison

Earth-Centred Inertial (ECI) Coordinate System

The coordinate time associated with an Earth-Centered Inertial (ECI) coordinate system, e.g. GCRS, is Geocentric Coordinate Time (TCG). Through terms of order $1/c^2$, the components of the metric tensor in this coordinate system are $-g_{00} = 1 - 2U/c^2$, $g_{0j} = 0$, and $g_{ij} = (1 + 2U/c^2)\,\delta_{ij}$, where U is the gravitational potential.

The elapsed TCG in the ECI coordinate system corresponding to the elapsed proper time during the transport of a clock between points A and B is given by:

$$\Delta t = \int_A^B \left(1 + \frac{1}{c^2}U + \frac{1}{2}\frac{1}{c^2}v^2\right) d\tau\ . \tag{11}$$

where U is the Earth's gravitational potential at the location of the clock excluding the centrifugal potential. v is the velocity of the clock with respect to the geoid. U may be expressed at radial distance r, geocentric latitude ϕ, and longitude λ as an expansion in spherical harmonics:

$$\begin{aligned} U(r,\phi,\lambda) &= \frac{GM_E}{r}\left[1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R_E}{r}\right)^n P_{nm}(\sin\phi)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\right] \\ &= \frac{GM_E}{r}\left[1 - \sum_{n=2}^{\infty} J_n\left(\frac{R_E}{r}\right)^n P_n(\sin\phi) - \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_E}{r}\right)^n P_{nm}(\sin\phi)\,(J_{mn}\cos m\lambda + K_{mn}\sin m\lambda)\right], \end{aligned} \tag{12}$$

where GM_E is the gravitational constant of the Earth and R_E is the equatorial radius of the Earth. The factors $P_n(\sin\phi)$ are the Legendre polynomials of degree n and the factors $P_{nm}(\sin\phi)$ are the associated Legendre functions of degree n and order m. The geocentric latitude ϕ_c is related to the geographic latitude ϕ_g by $\tan\phi_c = (1 - f^2)\tan\phi_g$, where f is the flattening.

Clock at Rest on the Geoid

For a clock at rest on the surface of the rotating Earth, it is necessary to account for the velocity of the clock $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ in the ECI coordinate system, where $\boldsymbol{\omega}$ is the angular velocity of the Earth and $\mathbf{r}$ is the position of the clock. Thus the coordinate time (TCG) elapsed as the clock records proper time $\Delta\tau$ is

$$\Delta t = \int_A^B \left(1 + \frac{1}{c^2}U + \frac{1}{2c^2}(\boldsymbol{\omega}\times\mathbf{r})^2\right) d\tau = \int_A^B \left(1 + \frac{1}{c^2}W_0\right) d\tau\,, \tag{13}$$

where W_0 is the gravity potential composed of the sum of the gravitational potential, U, and the rotational potential, $\tfrac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})^2$. As the gravity potential W_0 over the surface of the geoid is constant, it may be evaluated on the equator and is approximately given by:

$$W_0 \approx \frac{GM_E}{R_E}\left(1 + \frac{1}{2}J_2\right) + \frac{1}{2}\omega^2 R_E{}^2 \tag{14}$$

Time Transfer

When transferring time from point P to point Q by means of a clock the coordinate time elapsed during the motion of the clock is:

$$\Delta t = \int_P^Q \left[1 + \frac{U(\mathbf{r}) - \mathrm{U_g}}{c^2} + \frac{\mathbf{v}(\mathbf{r})^2}{2c^2}\right] d\tau, \tag{15}$$

where $U(\mathbf{r})$ is the gravitational potential at the location of the clock excluding the centrifugal potential, $\mathbf{v}(\mathbf{r})$ is the velocity of the clock as viewed in the geocentric non-rotating reference frame, and $\mathrm{U_g}$ is the potential at the geoid.

A Clock on an Earth Orbiting Satellite

For a clock carried on an Earth-orbiting satellite, the orbit may be regarded as Keplerian (unperturbed) in the first approximation. The potential at distance r from the Earth's centre is $U = GM_E/r$, and the coordinate time (TCG) elapsed as the clock records proper time $\Delta\tau$ is approximately

$$\Delta t = \int_A^B \left(1 - \frac{1}{c^2}\frac{GM_E}{2a} + \frac{1}{c^2}\frac{2GM_E}{r}\right) d\tau \approx \left(1 + \frac{3}{2}\frac{1}{c^2}\frac{GM_E}{a}\right)\Delta\tau + \frac{2}{c^2}\sqrt{GM_E\, a}\; e\sin E \tag{16}$$

where E is the eccentric anomaly determined from the mean anomaly by Kepler's equation, $M \equiv n\Delta t = E - e\sin E$, n being the mean motion given by $n \equiv 2\pi/T = \sqrt{\mathrm{GM}_E/a^3}$, T is the orbital period, and a is the orbital semi-major axis.

In order to compare the proper time of a clock on an Earth-orbiting satellite with the proper time of a clock at rest on the geoid this must be converted to TT. Thus, the interval of proper time recorded by a clock at rest on the geoid corresponding to the interval of proper time recorded by a clock on the satellite can be given by:

$$\Delta\tau_0 = \left[1 + \frac{3}{2}\frac{1}{c^2}\frac{GM_E}{a} - \frac{1}{\mathrm{c}^2}\frac{GM_E}{R_E}\left(1 + \frac{1}{2}J_2\right) - \frac{1}{2\mathrm{c}^2}\omega^2 R_E{}^2\right]\Delta\tau + \frac{2}{\mathrm{c}^2}\sqrt{GM_E\, a}\; e \sin E \qquad (17)$$

where J_2 is the Earth's second order oblateness coefficient. The second term is a correction for variation in velocity and potential due to the orbital eccentricity.

At the sub-nanosecond level of precision, it is necessary to take into account the orbital perturbations due to the harmonics of the Earth's gravitational potential, the tidal effects of the Moon and the Sun, and solar radiation pressure. At this level of precision, the J_2 perturbation produces variations in **r** and **v** that result in additional periodic effects on the order of 0.1 ns. Therefore, to fully account for the J_2 perturbation, it is necessary to perform a numerical integration of the orbit and a numerical integration of equation (16). The tidal effects of the Moon and the Sun and solar radiation pressure should also be considered.

For Low-Earth Orbits, both the zonal and tesseral gravitational harmonics are important. The usual eccentricity correction of equation (16) is no longer accurate. In this case, it is preferable to integrate the orbit and integrate equation (16) numerically including the higher order harmonics of the Earth's gravitational potential.

5 Earth-Centred Earth-Fixed (ECEF) Coordinate System

Through terms of order $1/\mathrm{c}^2$, the metric components are $-g_{00} = 1 - 2U/\mathrm{c}^2 - (\boldsymbol{\omega} \times \mathbf{r})^2/\mathrm{c}^2 = 1 - 2W_0/\mathrm{c}^2$, $g_{0j} = (\boldsymbol{\omega} \times \mathbf{r})_j/\mathrm{c}$, and $g_{ij} = \delta_{ij}$. In the rotating Earth-Centred Earth-Fixed (ECEF) coordinate system the coordinate time is equal to Terrestrial Time (TT), and the elapsed coordinate time accumulated during clock transport is:

$$\Delta t = \int_A^B \left[1 - \frac{\Delta U(\mathbf{r})}{\mathrm{c}^2} + \frac{1}{2\mathrm{c}^2}v^2\right] d\tau + \frac{1}{\mathrm{c}^2}\int_A^B (\boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{v}\, d\tau\,, \qquad (18)$$

where $\Delta U(\mathbf{r})$ is the gravitational potential difference (including the centrifugal potential) between the location of the clock at **r** and the geoid as viewed from an Earth-fixed coordinate system, in agreement with the convention (Resolution A4, IAU, 1992) that $\Delta U(\mathbf{r})$ is negative when the clock is above the geoid. When the height h of the clock is less than 24 km above the geoid, $\Delta U(\mathbf{r})$ may be approximated by gh, where g is the total acceleration due to gravity (including the rotational acceleration of the Earth) evaluated at the geoid. This approximation applies to all aerodynamic and earthbound transfers. When h is greater than 24 km, the potential difference $\Delta U(\mathbf{r})$ may be calculated to greater accuracy as follows:

$$\Delta U(\mathbf{r}) = \frac{GM_E}{r} + J_2 GM_E a_1^2\left(\frac{1 - 3\cos^2\theta}{2r^3}\right) + \omega^2 r^2 \sin^2\frac{\theta}{2} - \mathrm{U_g}. \qquad (19)$$

For time transfer at the 1 ns level of accuracy, this expression should not be used beyond a distance of about 50 000 km from the centre of the Earth.

The second integral in equation (18) is the Sagnac effect for a transported clock. This may be expressed:

$$\Delta t_{\text{Sagnac}} = \frac{\omega \mathrm{R}^2}{\mathrm{c}^2} \int_A^B \cos^2 \phi \, d\lambda \;= \frac{2\,\omega\, A_E}{\mathrm{c}^2}, \tag{20}$$

where R is the radius of the Earth, ϕ is the latitude, λ is the longitude, and where A_E is the projection onto the equatorial plane of the area swept out by the position vector with respect to the centre of the Earth (positive for the eastward direction and negative for the westward direction). The correction is positive for a clock traveling east and is negative for a clock traveling west.

Time Transfer

When transferring time from point A to point B by means of a portable clock, the coordinate time accumulated during transport is:

$$\Delta t = \int_A^B \left[1 + \frac{\Delta U(\mathbf{r})}{\mathrm{c}^2} + \frac{v^2(\mathbf{r})}{2\mathrm{c}^2}\right] d\tau + \frac{2\omega}{\mathrm{c}^2} A_E \tag{21}$$

where A_E is measured in an earth-fixed coordinate system. As the area A_E is swept, it is taken as positive when the projection of the path of the clock on the equatorial plane moves eastward. For time transfer at the 1 ns level of accuracy, this expression should not be used beyond a distance of about 50 000 km from the centre of the Earth. Beyond that distance the time transfer should be calculated in the Barycentric Coordinate System.

6 Barycentric Coordinate System

The interval of barycentric coordinate time (TCB) corresponding to an interval of proper time is:

$$\mathrm{TCB} \;=\; \int_{\tau_0}^{\tau} \left[1 + \frac{1}{\mathrm{c}^2} U_E(\mathbf{R}) + \frac{1}{2}\frac{1}{\mathrm{c}^2} |\dot{\mathbf{R}}|^2\right] d\tau \;+\; \frac{1}{\mathrm{c}^2} \int_{\tau_0}^{\tau} \left[U_{\text{ext}}(\mathbf{r}_E) + \frac{1}{2} v_E^2\right] d\tau \;+\; \frac{1}{\mathrm{c}^2}\, \mathbf{v}_E \cdot \mathbf{R}\Big|_{\tau_0}^{\tau}, \tag{22}$$

where **R** is the geocentric position of the clock, $\mathbf{r}_E$ is the barycentric position of the Earth's centre of mass, $U_E(\mathbf{R})$ is the Newtonian potential of the Earth, $U_{ext}(\mathbf{R})$ is the external Newtonian potential of all of the bodies in the solar system, apart from the Earth, evaluated at the clock, and $\mathbf{v}_\mathrm{E}$ is barycentric velocity of the Earth's centre of mass. Equation (22) applies to a clock anywhere in the vicinity of Earth, including a clock on a satellite or on the Earth's surface. For a clock at rest on the geoid, the first term is TCG. Therefore, to order $1/\mathrm{c}^2$ the coordinate time difference between TCB and TCG is:

$$\mathrm{TCB} - \mathrm{TCG} \;=\; \frac{1}{\mathrm{c}^2} \int_{t_0}^{t} \left[U_{\text{ext}}(\mathbf{r}_E) + \frac{1}{2} v_E^2\right] dt \;+\; \frac{1}{\mathrm{c}^2}\, \mathbf{v}_E(t) \cdot \mathbf{R}(t). \tag{23}$$

For time transfer between a solar system body and Earth, two transformations are required. The first transformation is from TT to TCB and the second is from TCB to time on the solar system body $\mathrm{T_{SSB}}$. As TCB is common to both transformations, it cancels out in computing the difference $\mathrm{T_{SSB}}$–TT. The transformation from TCB to TT is:

$$\mathrm{TCB} - \mathrm{TT} \approx (L_C + L_G)\,\mathrm{TCB} + P + \frac{1}{c^2}\, \mathbf{v}_E \cdot \mathbf{R}. \tag{24}$$

Similarly, the transformation from Barycentric Coordinate Time (TCB) to $\mathrm{T_{SSB}}$ is given by:

$$\mathrm{TCB} - \mathrm{T}_{SSB} \approx (L_{CSSB} + L_{SSB})\,\mathrm{TCB} + P + \frac{1}{\mathrm{c}^2}\, \mathbf{v}_{SSB} \cdot \mathbf{R}. \tag{25}$$

In the case of Mars, $L_{CSSB} = L_{CM}$ = 0.972 × 10^{-8} ≈ 0.84 ms/d, $L_{SSB} = L_M \equiv W_{0M}/c^2$ = 1.403 × 10^{-10} ≈ 12.1 μs/d, W_{0M} being the gravity potential on Mars.

7 Propagation of an Electromagnetic Signal

Earth-Centred Inertial (ECI) Coordinate System

For an electromagnetic signal propagating in an ECI coordinate system, the coordinate time of propagation (TCG) is:

$$\Delta t \approx \frac{1}{c}\int_{\text{path}} \sqrt{g_{ij}\, dx^i dx^j}\ . \tag{26}$$

In a first approximation the gravitational potential may be neglected. The metric is then $-g_{00} \approx 1$, $g_{0j} = 0$, and $g_{ij} \approx \delta_{ij}$, and the coordinate time (TT) is $(1 - L_G)\ \Delta t$. The right side of equation (26) is just ρ/c, where ρ is the Euclidian path length in the ECI system.

If the signal is transmitted at coordinate time t_T and is received at coordinate time t_R , the coordinate time (TCG) of propagation over the path is:

$$\Delta t = \frac{\rho}{c} = \frac{1}{c}\,|\,\mathbf{r}_R(t_R) - \mathbf{r}_T(t_T)\,| \approx \frac{1}{c}\,|\,\Delta\mathbf{r} + \mathbf{v}_R\ (t_R - t_T)\,| \approx \frac{1}{c}\,|\Delta\mathbf{r}| + \frac{1}{c^2}\,\Delta\mathbf{r}\cdot\mathbf{v}_R, \tag{27}$$

where, the transmitter has position $\mathbf{r}_T$ and the receiver has position $\mathbf{r}_R$ and velocity $\mathbf{v}_R$ and $\Delta\mathbf{r} \equiv \mathbf{r}_R(t_T) - \mathbf{r}_T(t_T)$ is the difference between the position of the receiver and the transmitter at the coordinate time of transmission t_T. The correction to the coordinate time due to the receiver velocity is:

$$\Delta t_{\text{vel}} \approx \Delta\mathbf{r}\cdot\mathbf{v}_R\,/\,c^2. \tag{28}$$

To account for the effect of the gravitational potential on an electromagnetic signal, it is necessary to include the potential in both the spatial and temporal parts of the metric. The gravitational time delay is:

$$\Delta t_{\text{delay}} = \frac{2\,GM_E}{c^3}\ln\left(\frac{R + r + \rho}{R + r - \rho}\right), \tag{29}$$

where R and r represent the geocentric radial distance of transmitter and receiver respectively. The coordinate time of propagation (TT) equivalent to the proper time interval recorded by a clock at rest on the geoid is:

$$(1 - L_G)\ \Delta t = \frac{\rho}{c} - L_G\,\frac{\rho}{c} + \frac{2\,GM_E}{c^3}\ln\left(\frac{R + r + \rho}{R + r - \rho}\right). \tag{30}$$

Earth-Centered Earth-Fixed (ECEF) Coordinate System

For an electromagnetic signal propagating in an ECEF coordinate system, the coordinate time of propagation (TCG) is:

$$\Delta t = \frac{1}{c}\int_{\text{path}} \sqrt{g_{ij}\, dx^i dx^j} + \frac{1}{c}\int_{\text{path}} g_{0j}\, dx^i. \tag{31}$$

The metric components are $-g_{00} = 1 - 2U/c^2$, $g_{0j} = (\boldsymbol{\omega} \times \mathbf{r})_j/c$, and $g_{ij} \approx \delta_{ij}$, where $\mathbf{r}$ is the position of a point on the signal path. The coordinate time (TT) is $(1 - L_G)\ \Delta t$. The coordinate time elapsed between emission and reception of an electromagnetic signal is then:

$$\Delta t = \frac{1}{c}\int_{\text{path}}\left[1+\frac{\Delta U(\mathbf{r})}{c^2}\right]dr + \frac{1}{c^2}\int_{\text{path}}(\boldsymbol{\omega}\times\mathbf{r})\,d\mathbf{r}, \tag{32}$$

where dr is the increment of standard length, or proper length, along the transmission path, $\Delta U(\mathbf{r})$ is the potential at the point, $\mathbf{r}$, on the transmission path less the potential at the geoid (see equation (19)), as viewed from an Earth-fixed coordinate system.

Neglecting $\Delta U(\mathbf{r})$, the first term of equation (32) is ρ'/c, where ρ' is the Euclidean path length in the ECEF coordinate system. If the transmitter has position $\mathbf{r}_T$ and the receiver has position $\mathbf{r}_R$ and velocity $\mathbf{v}'_R$, then:

$$\frac{\rho'}{c} = \frac{1}{c}\,|\,\mathbf{r}_R(t_R)-\mathbf{r}_T(t_T)\,| \approx \frac{1}{c}\,|\,\Delta\mathbf{r}+\mathbf{v}'_R\;(t_R-t_T)\,| \approx \frac{1}{c}\,|\Delta\mathbf{r}|+\frac{1}{c^2}\,\Delta\mathbf{r}\cdot\mathbf{v}'_R. \tag{33}$$

where $\Delta\mathbf{r} \equiv \mathbf{r}(t_T) - \mathbf{r}_T(t_T)$.

The second term of equation (32) is the Sagnac effect:

$$\Delta t_{\text{Sagnac}} \approx \frac{1}{c^2}\int_A^B(\boldsymbol{\omega}\times\mathbf{r})\cdot d\mathbf{r} = \frac{2\,\omega\,A_E}{c^2}\quad, \tag{34}$$

where A_E is the projection onto the equatorial plane of the area formed by the center of rotation and the endpoints of the signal path. The area is positive when the signal path has an eastward component.

Barycentric Coordinate System

If an electromagnetic signal is sent in a barycentric coordinate system with Cartesian coordinates (x, y, z) from a transmitter at point $(-a_T, b, 0)$ to a receiver at point $(a_R, b, 0)$ along the approximately straight line path $y = b$ (neglecting gravitational deflection), where b is the distance of closest approach to the Sun. The coordinate time of propagation (TCB) is:

$$\Delta t \approx \frac{1}{c}\int_{\text{path}}\sqrt{\frac{g_{ij}}{-g_{00}}\,dx^i dx^j} \approx \frac{1}{c}\int_{-a_T}^{a_R}\left(1+\frac{2}{c^2}U_S\right)dx = \frac{1}{c}\int_{-a_T}^{a_R}\left(1+\frac{1}{c^2}\,\frac{2\,GM_S}{\sqrt{x^2+b^2}}\right)dx, \tag{35}$$

where U_S is the gravitational potential of the Sun. Therefore,

$$\Delta t = \frac{1}{c}(a_T+a_R)+2\frac{GM_S}{c^3}\ln\frac{a_R+\sqrt{a_R{}^2+b^2}}{-a_T+\sqrt{a_T{}^2+b^2}}. \tag{36}$$

Scaled to the Terrestrial Time (TT) of propagation recorded by a clock on the geoid,

$$\Delta t' = (1-L_B)\,\Delta t = \frac{1}{c}(1-L_B)(a_T+a_R)+2\frac{GM_S}{c^3}\ln\frac{a_R+\sqrt{a_R{}^2+b^2}}{-a_T+\sqrt{a_T{}^2+b^2}}. \tag{37}$$

8 Examples

Due to relativistic effects, a clock at an elevated location will appear to be higher in frequency and will differ in normalized rate from TAI by $-\Delta U/c^2$. Near sea level this is given by $-g(\phi)h/c^2$, where ϕ is the geographical latitude, $g(\phi)$ is the total acceleration at sea level (gravitational and centrifugal) = $(9.780 + 0.052 \sin^2 \varphi)$ m/s^2, and h is the distance above sea level.

If a clock is moving relative to the Earth's surface with speed V, which may have a component V_E in the direction to the East, the normalized difference in frequency of the moving clock relative to that of a clock at rest at sea level is:

$$\Delta f = -\frac{1}{2}\frac{V^2}{c^2} + \frac{g(\phi)h}{c^2} - \frac{1}{c^2}\omega r \cos\phi\, V_E, \tag{38}$$

where r is the distance of the clock from the center of the Earth.

The choice of a coordinate frame is purely a discretionary one, but to define coordinate time, a specific choice must be made. It is recommended that for terrestrial use an Earth-fixed frame be chosen. In this frame, when a clock B is synchronized with a clock A (both clocks being stationary on the Earth) by a radio signal travelling from A to B, these two clocks differ in coordinate time by:

$$t_B - t_A = -\frac{\omega}{c^2}\int_{path} r^2 \cos^2\phi d\lambda, \tag{39}$$

where ϕ is latitude, λ is East longitude and *path* refers to the path over which the radio signal travels from A to B.

If the two clocks are synchronized by a portable clock, they will differ in coordinate time by:

$$t_B - t_A = \int_{path}\left[\frac{\Delta U(\mathbf{r})}{c^2} - \frac{V^2}{2c^2}\right]dr - \frac{\omega}{c^2}\int_{path} r^2 \cos^2\phi\, d\lambda, \tag{40}$$

where V is the portable clock's ground speed, and *path* refers to the path over which the clock travels from A to B.

9 Glossary

A_E	equatorial projection of the area swept out during the time transfer by the vector **r** as its terminus moves
BCRS	Barycentric Celestial Reference System
c	speed of light = 2.997 924 58 × 10^8 m/s
ECEF	Earth-centred, Earth-fixed
ECI	Earth centred inertial
f	flattening of the Earth = 1/298.257 223 563
GCRS	Geocentric celestial reference system, an ECI system of geocentric space-time coordinates
GM_E	gravitational constant of the Earth = 398 600 km^3/s^2
GTRS	Geocentric terrestrial reference system, an ECEF coordinate system
J_{mn}:	Coefficients of spherical harmonics describing the oblateness of the Earth. J_2 (m = 2, n = 0) is the most significant and can be defined in terms of the polar (C) and equatorial (A) moments of inertia of the Earth: $J_2 = \frac{C-A}{MR^2}$, where M and R are the mass and radius of the Earth respectively. $J_2 = +1.083 \times 10^{-3}$
L_B	$1 - (1 - L_G)(1 - L_C)$
L_c	$1 - \langle d\mathrm{TCG}/d\mathrm{TCB}\rangle$, $\langle d\mathrm{TCG}/d\mathrm{TCB}\rangle$ being an average offset in rate
L_G	$1 - d\mathrm{TT}/d\mathrm{TCG}$, a defined constant ≡ 6.969 290 134 × 10^{-10}

r	vector whose origin is at the centre of the Earth and whose terminus moves with the clock
r	magnitude of the vector **r**
R_E	equatorial radius of the Earth = 6 378 136 km
t	coordinate time, the independent variable in the equations of motion of material bodies and in the equations of propagation of electromagnetic waves
TAI	International atomic time
TCB	Barycentric coordinate time
TCG	Geocentric coordinate time
TDB	Barycentric dynamical time
TT	Terrestrial time
UTC	Coordinated universal time
U	gravitational potential
U_g	potential (gravitational and centrifugal) at the geoid = 62 636.86 km^2/s^2.
v	velocity of the clock with respect to the ground
W	the gravity potential composed of *U*, and the rotational potential, $\frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})^2$
s_{ij}	Kronecker delta function: $\delta_{ij} = 0$ if $i \neq j$; $\delta_{ij} = 1$ if $i = j$
$\Delta U(\mathbf{r})$:	gravitational potential difference (including the centrifugal potential) between the location of the clock at **r** and the geoid as viewed from an Earth-fixed coordinate system, in agreement with the convention (Resolution A4, IAU, 1992) that $\Delta U(\mathbf{r})$ is negative when the clock is above the geoid
θ	colatitude
λ	longitude
ϕ	latitude
ϕ_c	geocentric latitude ϕ_c
ϕ_g	geographic latitude ϕ_g
τ	proper time as measured in the "rest frame" of the clock; that is, in the reference frame travelling with the clock.
ω	angular velocity of rotation of the Earth = 7.292 115 × 10^{-5} rad/s

References

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