

International Telecommunication Union

ITU-R
Radiocommunication Sector of ITU

Recommendation ITU-R TF.2018
(08/2012)

**Relativistic time transfer in the vicinity of
the Earth and in the solar system**

TF Series
Time signals and frequency standards emissions

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Note: This ITU-R Recommendation was approved in English under the procedure detailed in Resolution ITU-R 1.

Electronic Publication
Geneva, 2012

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RECOMMENDATION ITU-R TF.2018

Relativistic time transfer in the vicinity of the Earth and in the solar system

(2012)

Scope

The purpose of this Recommendation is to establish common conventional algorithms and procedures to be used in comparing clocks on the surface of the Earth and on platforms far from the Earth but within the solar system. These expressions are explicitly determined in the general relativity theory that is presently accepted to form the basis of space-time reference systems. It is envisioned that these algorithms and procedures would be used for comparisons of clocks on Earth satellites, interplanetary spacecraft, and on the surfaces of solar system bodies.

The ITU Radiocommunication Assembly,

considering

- a) that it is desirable to maintain coordination of standard time and frequency on platforms operating in the vicinity of the Earth and in the solar system;
- b) that accurate means of transferring time and frequency are required to meet the future needs of timekeeping, navigation, science, and communication systems in the vicinity of the Earth and in the solar system;
- c) that clocks are subject to path-dependent time and frequency variations due to their motion and to the gravitational potential in which they operate;
- d) that the conceptual foundation for the transfer of time and frequency should be clearly outlined;
- e) that procedures for the transfer of time and frequency in the vicinity of the Earth and across celestial bodies and spacecraft in the solar system require the use of mathematical algorithms that account for relativistic effects;
- f) that requirements for precision and accuracy for the transfer of time and frequency in the vicinity of the Earth and in the solar system depend on the specific application,

recommends

that the mathematical algorithms that account for relativistic effects in the transfer of time and frequency as provided in Annex 1 should be used, as appropriate.

Annex 1

Objective

The purpose of this Recommendation is to raise the level of awareness for the need to address the effects of relativity in timekeeping, navigation, science, and communication systems. The Recommendation recalls the basic concepts and procedures to be applied in the analysis of such systems. No attempt is made to describe the details of any particular system. Rather, it is intended that the information presented here may serve as a convenient reference and a point of departure for specific applications.

One important application of this Recommendation is the comparison of the times registered by clocks on spacecraft in orbit around the Earth, in interplanetary space, and on planetary surfaces with the times recorded by clocks on Earth. An appropriate timescale for terrestrial measurements is Coordinated Universal Time (UTC). Therefore, the objective might be to relate the times registered by clocks, wherever they may be in the vicinity of the Earth and in the solar system, with the times of clocks on Earth that measure UTC.

The following discussion is based on the *IERS Conventions (2010)*, the ITU-R Handbook on *Satellite Time and Frequency Transfer and Dissemination (2010)*, Nelson, *Metrologia (2011)*, and Petit and Wolf, *Metrologia (2005)*. Users may consult those publications and references cited therein for further details.

Relativistic framework

The relativistic framework for space-time reference systems has been defined by Resolutions of international scientific organizations. The most important are:

- 1) IAU Resolution A4 (1991) defines the Geocentric Celestial Reference System (GCRS) and the Barycentric Celestial Reference System (BCRS) and their time coordinates. IAU Resolution B1 (2000) further refines the BCRS definition.
- 2) IUGG Resolution 2 (2007) defines the Geocentric Terrestrial Reference System (GTRS), along with the International Terrestrial Reference System (ITRS).

The nomenclature used in this document follows the one used in the past ITU-R Recommendations and may be related to the IAU/IUGG framework as follows: in this Recommendation the GCRS is termed the Earth-Centered Inertial (ECI) coordinate system, the GTRS (in practice, the ITRS) is termed the Earth-Centered Earth-Fixed (ECEF) coordinate system, and the BCRS is termed the barycentric coordinate system.

Definitions

Proper time

Proper time τ is the actual reading of a clock or the local time in the clock's own frame of reference.

Coordinate time

Coordinate time t is the independent variable in the equations of motion of material bodies and in the equations of propagation of electromagnetic waves. It is a mathematical coordinate in the four-dimensional space-time of the coordinate system. For a given event, the coordinate time has the same value everywhere. Coordinate times are not measured; rather, they are computed from the proper times of clocks.

Space-time interval

The relation between coordinate time and proper time depends on the clock's position and state of motion in its gravitational environment and is derived by integration of the space-time interval. In the comparison of the proper times of two clocks, the coordinate time is ultimately eliminated. Thus the relativistic transfer of time between clocks is independent of the coordinate system. The coordinate system may be chosen arbitrarily on the basis of convenience.

In general, the space-time interval is described by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} c^2 dt^2 + 2 g_{0j} c dt dx^j + g_{ij} dx^i dx^j \quad (1)$$

where:

$g_{\mu\nu}$: components of the metric.

A Greek index assumes the range 0, 1, 2, 3 and a Latin index assumes the range 1, 2, 3. A repeated index implies summation on that index. The metric depends upon the gravitational potentials and upon the angular velocity and linear acceleration of the reference frame. Upon a transformation of the coordinates, the space-time interval remains invariant. Thus the metric $g_{\mu\nu}$ transforms as a second order covariant tensor.

The general expression for the relationship between proper time τ and the coordinates of the chosen coordinate system, comprising the coordinate time $x^0 \equiv ct$ and the spatial coordinates x^i , is given by:

$$ds^2 = g_{00} c^2 dt^2 + 2 g_{0j} c dt dx^j + g_{ij} dx^i dx^j = -c^2 d\tau^2 \quad (2)$$

where:

τ : the proper time.

Thus $dt = d\tau$ for a clock at rest in an inertial frame of reference, for which $dx^i = 0$ and $-g_{00} = 1$, $g_{0j} = 0$, and $g_{ij} = \delta_{ij}$. The elapsed coordinate time corresponding to the measured proper time as registered by a clock along a path between points A and B is:

$$\Delta t = \pm \int_A^B \frac{1}{\sqrt{-g_{00}}} \sqrt{1 + \frac{1}{c^2} \left(g_{ij} + \frac{g_{0i} g_{0j}}{-g_{00}} \right) \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} d\tau + \frac{1}{c} \int_A^B \frac{g_{0j}}{-g_{00}} \frac{dx^j}{d\tau} d\tau \quad (3)$$

For an electromagnetic signal, the space-time interval is:

$$ds^2 = g_{00} c^2 dt^2 + 2 g_{0j} c dt dx^j + g_{ij} dx^i dx^j = 0 \quad (4)$$

The speed of light is c in every inertial frame of reference. The elapsed coordinate time of propagation along a path between points A and B is:

$$\Delta t = \pm \frac{1}{c} \int_A^B \frac{1}{\sqrt{-g_{00}}} \sqrt{\left(g_{ij} + \frac{g_{0i} g_{0j}}{-g_{00}} \right) dx^i dx^j} + \frac{1}{c} \int_A^B \frac{g_{0j}}{-g_{00}} dx^j \quad (5)$$

The expression $\gamma_{ij} \equiv g_{ij} + g_{0i} g_{0j} / (-g_{00})$ represents the metric of three-dimensional space and $d\rho = \sqrt{\gamma_{ij} dx^i dx^j}$ represents the increment of three-dimensional distance.

Time scales

Atomic time scales

The fundamental scale of time based on atomic clocks is International Atomic Time (TAI), which is calculated at the BIPM from a weighted average of the readings of atomic clocks in timing laboratories distributed around the world. It is a continuous reference timescale without steps.

The atomic timescale for civil timekeeping is Coordinated Universal Time (UTC), which differs from TAI by an integral number of seconds. In 2011, $UTC = TAI - 34$ s. UTC is disseminated every month in BIPM *Circular T* in the form of the differences from individual laboratory realizations $UTC(k)$.

Coordinate time scales

Geocentric Coordinate Time (TCG) is the coordinate time in a coordinate system with origin at the Earth's center (ECI or ECEF).

Terrestrial Time (TT) is another coordinate time that is rescaled from TCG so that it has approximately the same rate as the proper time of a clock at rest on the geoid. The geoid is the surface of constant gravity potential, which is closely approximated by mean sea level. The relationship between TCG and TT is defined such that $dTT/dTCG \equiv 1 - L_G$, where $L_G \equiv 6.969\,290\,134 \times 10^{-10} \approx 60.2$ $\mu\text{s/d}$ as discussed below following equation (18). The value of L_G is a defined constant. Consequently,

$$TCG - TT = L_G TCG = \frac{L_G}{1 - L_G} (TT - TT_0)$$

$$TCG - TT = L_G (TCG - TCG_0) = \frac{L_G}{1 - L_G} (TT - TT_0) \quad (6)$$

where:

TCG_0 and TT_0 : correspond to JD 2443144.5 TAI (1977 January 1, 0h). A practical realization of TT is

$$TT = TAI + 32.184 \text{ s.} \quad (7)$$

Barycentric Coordinate Time (TCB) is the coordinate time in a coordinate system with origin at the solar system barycenter. The coordinate time difference between TCB and TCG is a transformation that depends on both time and position. To order $1/c^2$

$$TCB - TCG = \frac{1}{c^2} \int_{t_0}^t \left(U_{ext}(\mathbf{r}_E) + \frac{1}{2} v_E^2 \right) dt + \frac{1}{c^2} \mathbf{v}_E(t) \cdot \mathbf{R}(t) \quad (8)$$

where:

$\mathbf{R}(t) = \mathbf{x} - \mathbf{x}_E$: time-dependent position vector with respect to the geocenter

\mathbf{x} : barycentric position of the observer and \mathbf{x}_e and \mathbf{v}_e denote the barycentric position and velocity of the Earth's center of mass.

This equation may be expressed in the form

$$TCB - TCG = L_C \times (TCB - TCB_0) + P(TCB) - P(TCB_0) + \frac{1}{c^2} \mathbf{v}_E(\mathbf{x} - \mathbf{x}_E) \quad (9)$$

where:

$$L_C = 1.480\,826\,867\,41 \times 10^{-8} \approx 1.28 \text{ ms/d.}$$

In this expression, P represents a series of periodic terms. The last term is diurnal at the surface of the Earth, with an amplitude smaller than 2.1 μs .

An alternative formulation to equation (9) is (*IERS Conventions (2010)*, Chapter 10)

$$TCB - TCG = \frac{L_C \times (TT - TT_0) + P(TT) - P(TT_0)}{1 - L_B} + \frac{1}{c^2} \mathbf{v}_E(\mathbf{x} - \mathbf{x}_E) \quad (10)$$

where:

$$TT \text{ and } L_B \equiv 1.550\,519\,768 \times 10^{-8} \approx 1.34 \text{ ms/d is the time argument.}$$

The value of L_B is a defined constant.

Periodic terms denoted by $P(TT)$ have a maximum amplitude of around 1.6 ms and can be evaluated by the “FB” analytical model (Fairhead and Bretagnon, 1990). Alternatively, $P(TT) - P(TT_0)$ may be provided by a numerical time ephemeris such as TE405 (Irwin and Fukushima, 1999), which provides values with an accuracy of ± 0.1 ns from 1600 to 2200. A series, HF2002, providing the value of $L_C (TT - TT_0) + P(TT) - P(TT_0)$ as a function of TT over the years 1600-2200 has been fit (Harada and Fukushima, 2003) to TE405. This fit differs from TE405 by less than 3 ns over the years 1600-2200 with an rms error of ± 0.5 ns.

The difference between TCB and TT is:

$$TCB - TT = (TCB - TCG) + TCG - TT = L_B TCB + (1 - L_G) \left(P + \frac{1}{c^2} \mathbf{v}_E \cdot \mathbf{R} \right) \quad (11)$$

The transformation from TCB to TCG consists of an average offset in rate $\langle dTCG/dTCB \rangle \equiv 1 - L_C$ and periodic terms. The transformation from TCG to TT is an exact offset in rate $dTT/dTCG \equiv 1 - L_G$. Thus the transformation from TCB to TT has an average offset in rate.

$$\langle dTT/dTCB \rangle = (dTT/dTCG) \langle dTCG/dTCB \rangle = (1 - L_G)(1 - L_C) \quad (12)$$

From the definition of L_B , $(1 - L_G)(1 - L_C) \approx (1 - L_B)$, so that equation (12) may be expressed as $\langle dTT/dTCB \rangle = (1 - L_B)$ to within a few parts in 10^{18} .

Similarly to TT , Barycentric Dynamical Time (TDB) is another coordinate time in a barycentric system rescaled to have approximately the same rate as TT . The relationship between TCB and TDB is defined such that $dTDB/dTCB \equiv 1 - L_B$.

Relativistic effects on clocks

In the following discussion, the transformation between the proper time of an ideal clock (one that exactly realizes the SI second) and the coordinate time in the geocentric and barycentric coordinate systems is considered.

Earth-Centered Inertial coordinate system

The coordinate time associated with an Earth-Centered Inertial (ECI) coordinate system is Geocentric Coordinate Time (TCG). Through terms of order $1/c^2$, the components of the metric tensor in this coordinate system are $-g_{00} = 1 - 2U/c^2$, $g_{0j} = 0$, and $g_{ij} = (1 + 2U/c^2) \delta_{ij}$, where U is the gravitational potential. The elapsed TCG in the ECI coordinate system corresponding to the elapsed proper time as registered by a clock moving along a path between points A and B with velocity v is given by:

$$\Delta t = \int_A^B \left(1 + \frac{1}{c^2} U + \frac{1}{2} \frac{1}{c^2} v^2 \right) d\tau \quad (13)$$

The Earth's potential at radial distance r , geocentric latitude ϕ , and longitude λ may be expressed as an expansion in spherical harmonics as:

$$\begin{aligned} U(r, \phi, \lambda) &= \frac{GM}{r} \left\{ 1 + \sum_{n=2}^{\infty} \sum_{m=0}^n \left(\frac{R_E}{r} \right)^n P_{nm}(\sin \phi) (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \right\} \\ &= \frac{GM}{r} \left\{ 1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_E}{r} \right)^n P_n(\sin \phi) - \sum_{n=2}^{\infty} \sum_{m=1}^n \left(\frac{R_E}{r} \right)^n P_{nm}(\sin \phi) (J_{mn} \cos m\lambda + K_{mn} \sin m\lambda) \right\} \quad (14) \end{aligned}$$

where:

GM : the gravitational constant of the Earth

R_E : the equatorial radius of the Earth

The factors $P_n(\sin \phi)$: the Legendre polynomials of degree n

the factors $P_{nm}(\sin \phi)$: the associated Legendre functions of degree n and order m .

The geocentric latitude ϕ is related to the geographic latitude φ by $\tan \phi = (1 - f^2) \tan \varphi$, where f is the flattening.

For practical applications it may be sufficient to include only the first oblateness correction and approximate the gravitational potential as

$$U = \frac{GM}{r} - J_2 \left(\frac{R_E}{r} \right)^2 P_2(\sin \phi) = \frac{GM}{r} + J_2 \left(\frac{R_E}{r} \right)^2 (1 - 3 \sin^2 \phi) \quad (15)$$

1) Clock at rest on the geoid

For a clock at rest on the surface of the rotating Earth, it is necessary to account for the velocity of the clock $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ in the ECI coordinate system, where $\boldsymbol{\omega}$ is the angular velocity of the Earth and \mathbf{r} is the position of the clock. Thus TCG elapsed as the clock records proper time $\Delta\tau$ is:

$$\Delta t = \int_A^B \left(1 + \frac{1}{c^2} U + \frac{1}{2} \frac{1}{c^2} (\boldsymbol{\omega} \times \mathbf{r})^2 \right) d\tau = \int_A^B \left(1 + \frac{1}{c^2} W \right) d\tau \quad (16)$$

where:

$$W = U + \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 = U + \frac{1}{2}\omega^2 r^2 \cos^2 \phi: \quad \text{the gravity potential.}$$

As the gravity potential W_0 over the surface of the geoid is constant, it may be evaluated on the equator and is approximately given by:

$$W_0 \approx \frac{GM}{R_E} \left(1 + \frac{1}{2} J_2 \right) + \frac{1}{2} \omega^2 R_E^2 \quad (17)$$

The current best estimate of W_0 is $6.2636856 \times 10^7 \text{ m}^2/\text{s}^2$. According to equation (16), the TCG in the ECI coordinate system that corresponds to the proper time $\Delta\tau_0$ measured by a clock at rest on the geoid is:

$$\Delta t \equiv TCG = (1 + W_0 / c^2) \Delta\tau_0 \cong (1 + L_G) \Delta\tau_0 \quad (18)$$

where:

$$L_G \equiv 6.969\,290\,134 \times 10^{-10}.$$

By convention, the value of L_G is a defined constant. It represents the best available value of W_0 / c^2 at the time of its definition in 2000. TT is obtained by rescaling TCG by the factor $1 - L_G$. Thus:

$$\Delta t' \equiv TT = (1 - L_G) TCG \quad (19)$$

It follows that $TT = (1 - L_G)(1 + L_G) \Delta\tau_0 \cong \Delta\tau_0$ to within a few parts in 10^{18} .

2) Clock on a satellite

For a clock on an Earth orbiting satellite, the orbit may be regarded as Keplerian (unperturbed) in the first approximation. The potential at distance r from the Earth's center is approximately $U = GM / r$. Thus the increment of TCG is:

$$\Delta t = \int_A^B \left(1 + \frac{1}{c^2} \frac{GM}{r} + \frac{1}{2} \frac{1}{c^2} v^2 \right) d\tau \quad (20)$$

The satellite velocity v is determined by conservation of energy per unit mass \mathcal{E} :

$$\mathcal{E} = \frac{1}{2} v^2 - U = \frac{1}{2} v^2 - \frac{GM}{r} = -\frac{GM}{2a} \quad (21)$$

where:

a : the orbital semimajor axis.

Therefore, to this order, the elapsed coordinate time is:

$$\Delta t = \int_A^B \left(1 - \frac{1}{c^2} \frac{GM}{2a} + \frac{1}{c^2} \frac{2GM}{r} \right) d\tau = \left(1 - \frac{1}{c^2} \frac{GM}{2a} \right) \Delta\tau + \frac{2GM}{c^2} \int_{t_0}^t \frac{1}{r} dt \quad (22)$$

In the second integral $d\tau$ is replaced by dt as this term is a relativistic correction of order $1/c^2$. For a Keplerian orbit the radial distance is $r = a(1 - e \cos E)$, where e is the orbital eccentricity and E is the eccentric anomaly. The eccentric anomaly is determined from the mean anomaly by Kepler's equation, $M \equiv n \Delta t = E - e \sin E$, where the mean motion is $n \equiv 2\pi/T = \sqrt{GM/a^3}$ and T is the orbital period. Therefore, the TCG elapsed as the clock records proper time $\Delta\tau$ is approximately:

$$\Delta t = \int_A^B \left(1 - \frac{1}{c^2} \frac{GM}{2a} + \frac{1}{c^2} \frac{2GM}{r} \right) d\tau = \left(1 + \frac{3}{2} \frac{1}{c^2} \frac{GM}{a} \right) \Delta\tau + \frac{2}{c^2} \sqrt{GM a} e \sin E \quad (23)$$

The second term is a periodic correction due to the orbital eccentricity that causes a residual variation in distance and velocity given by:

$$\Delta t_{\text{eccentricity}} = \frac{2}{c^2} \sqrt{GM a} e \sin E = \frac{2}{c^2} \mathbf{v} \cdot \mathbf{r} \quad (24)$$

In this expression it is assumed that Keplerian (unperturbed) elements are used.

To compare the proper time of a clock on a satellite with the proper time of a clock at rest on the geoid, it is necessary to convert from *TCG* to *TT*. By equations (19) and (20), one obtains (*TT*):

$$\Delta t' = (1 - L_G) \Delta t = \int_A^B \left\{ 1 + \frac{1}{c^2} (U - W_0) + \frac{1}{2} \frac{1}{c^2} v^2 \right\} d\tau \quad (25)$$

Therefore, as $\Delta t' \approx \Delta\tau_0$, the interval of proper time recorded by a clock at rest on the geoid corresponding to the interval of proper time recorded by a clock on the satellite is:

$$\Delta\tau_0 = \left[1 + \frac{3}{2} \frac{1}{c^2} \frac{GM}{a} - \frac{1}{c^2} W_0 \right] \Delta\tau + \frac{2}{c^2} \sqrt{GM a} e \sin E \quad (26)$$

where:

GM : the gravitational constant of the Earth

R_E : the equatorial radius of the Earth.

At the sub-nanosecond level of precision, it is necessary to take into account the orbital perturbations due to the harmonics of the Earth's gravitational potential, the tidal effects of the Moon and the Sun, and solar radiation pressure. At this level of precision, the J_2 perturbation produces variations in \mathbf{r} and \mathbf{v} that result in additional periodic effects on the order of 0.1 ns.

To fully account for the J_2 perturbation in the potential of equation (15), it is necessary to perform a numerical integration of the orbit and a numerical integration of equation (20). The tidal effects of the Moon and the Sun and solar radiation pressure should also be considered.

For Low-Earth orbits, both the zonal and tesseral gravitational harmonics are important. The usual eccentricity correction of equation (24) is no longer accurate. In this case, it is preferable to integrate the orbit and integrate equation (20) numerically including the higher order harmonics of the Earth's gravitational potential.

Earth-Centered Earth-Fixed coordinate system

Through terms of order $1/c^2$, the metric components are $-g_{00} = 1 - 2U/c^2 - (\boldsymbol{\omega} \times \mathbf{r})^2/c^2 = 1 - 2W/c^2$, $g_{0j} = (\boldsymbol{\omega} \times \mathbf{r})_j/c$, and $g_{ij} = \delta_{ij}$. In the rotating Earth-Centered Earth-Fixed (ECEF) coordinate system using coordinate time TT , the elapsed coordinate time is:

$$\Delta t' = \int_A^B \left(1 - \frac{1}{c^2} g h + \frac{1}{2} \frac{1}{c^2} v'^2 \right) d\tau + \frac{1}{c^2} \int_A^B (\boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{v}' d\tau \quad (27)$$

where:

- h : the height above the geoid
- g : the local acceleration of gravity
- v' : the velocity of the clock with respect to the geoid.

It is assumed that h is small. For high accuracy, the variation of g with latitude and elevation should be taken into account.

The second integral is the Sagnac effect for a transported clock. This may be expressed as:

$$\Delta t_{Sagnac} = \frac{1}{c^2} \int_A^B (\boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{v}' d\tau = \frac{1}{c^2} \int_A^B (\omega R \cos \phi) (v' \cos \theta) d\tau = \frac{\omega R^2}{c^2} \int_A^B \cos^2 \phi d\lambda \quad (28)$$

or as:

$$\Delta t_{Sagnac} = \frac{\omega R^2}{c^2} \int_A^B \cos^2 \phi d\lambda = \frac{2\omega A}{c^2} \quad (29)$$

where:

- R : the radius of the Earth
- ϕ : the latitude
- λ : the longitude
- $v' \cos \theta$: the eastward component of the velocity
- A : the projection onto the equatorial plane of the area swept out by the position vector with respect to the center of the Earth (positive for the eastward direction and negative for the westward direction).

The correction is positive for a clock travelling east and is negative for a clock travelling west.

Barycentric coordinate system

The interval of barycentric coordinate time (TCB) corresponding to the proper time interval $\Delta\tau = \tau - \tau_0$ is:

$$TCB = \int_{\tau_0}^{\tau} \left(1 + \frac{1}{c^2} U_E(\mathbf{R}) + \frac{1}{2} \frac{1}{c^2} |\mathbf{R}|^2 \right) d\tau + \frac{1}{c^2} \int_{\tau_0}^{\tau} \left(U_{ext}(\mathbf{r}_E) + \frac{1}{2} v_E^2 \right) d\tau + \frac{1}{c^2} \mathbf{v}_E \cdot \mathbf{R} \Big|_{\tau_0}^{\tau} \quad (30)$$

where:

- $U_E(\mathbf{r})$: the Newtonian potential of the Earth
 $U_{\text{ext}}(\mathbf{r})$: the external Newtonian potential of all of the solar system bodies apart from the Earth.

Coordinate system of solar system bodies

For clock comparisons between a solar system body M and Earth, several transformations are required. The proper times of the clocks have to be transformed to TT for the Earth-bound and to TM for the M-bound) clock. Then the first transformation is from TT to TCB and the second is the corresponding transformation from TCB to TM . The coordinate transformations are:

$$TCB - TT = (L_C + L_G) TCB + P + \mathbf{v}_E \cdot \mathbf{R} / c^2 \quad (31)$$

and

$$TCB - TM = (L_{CM} + L_M) TCB + P + \mathbf{v}_M \cdot \mathbf{R} / c^2 \quad (32)$$

In these equations the periodic terms P and vector position R each apply to the Earth and the planetary body M, respectively. The difference between TM and TT is:

$$TM - TT = (TCB - TT) - (TCB - TM) \quad (33)$$

As an example, in the case of Mars, $L_{CM} = 0.972 \times 10^{-8} \approx 0.84$ ms/d, $L_M = 1.403 \times 10^{-10} \approx 12.1$ μ s/d. The drift rate is 0.49 ms/d. The amplitudes of the periodic terms are 1.7 ms at the Earth orbital period (365.2422 d) and 11.4 ms at the Mars orbital period (687 d).

Propagation of an electromagnetic signal

This section deals with the computation of the coordinate time of propagation of an electromagnetic signal when the transmitter and receiver positions are both given, as expressed in the ECI, ECEF, and barycentric coordinate systems.

These equations apply in all cases. In particular, they must be used when setting the parameters of clocks on satellites that are steered to clocks on the Earth.

Earth-Centered Inertial coordinate system

When considering the computation in an Earth-Centered Inertial (ECI) coordinate system, the coordinate time of propagation (TCG) may be considered as the sum of a geometric part and a gravitational part. The geometric part is:

$$\Delta t \approx \frac{1}{c} \int_{\text{path}} \sqrt{g_{ij}} dx^i dx^j = \frac{\rho}{c} \quad (34)$$

where:

- $g_{ij} \approx \delta_{ij}$ and
 ρ : the geometric path length.

If the signal is transmitted at coordinate time t_T and is received at coordinate time t_R , the TCG of propagation over the path is:

$$\Delta t = \frac{\rho}{c} = \frac{1}{c} |\mathbf{r}_R(t_R) - \mathbf{r}_T(t_T)| \approx \frac{1}{c} |\Delta \mathbf{r} + \mathbf{v}_R(t_R - t_T)| \approx \frac{1}{c} |\Delta \mathbf{r}| + \frac{1}{c^2} \Delta \mathbf{r} \cdot \mathbf{v}_R \quad (35)$$

where, the transmitter has position \mathbf{r}_T and the receiver has position \mathbf{r}_R and velocity \mathbf{v}_R and $\Delta \mathbf{r} \equiv \mathbf{r}_R(t_T) - \mathbf{r}_T(t_T)$ is the difference between the position of the receiver and the transmitter at the coordinate time of transmission t_T . The correction to the coordinate time due to the receiver velocity is:

$$\Delta t_{\text{vel}} \approx \Delta \mathbf{r} \cdot \mathbf{v}_R / c^2 \quad (36)$$

Note that additional terms of order $1/c^3$ may amount to several picoseconds, depending on the configuration.

To consider the effect of the gravitational potential on an electromagnetic signal, it is necessary to include the potential in both the spatial and temporal parts of the metric. The components of the metric are $-g_{00} = 1 - 2U/c^2$, $g_{0j} = 0$, and $g_{ij} = (1 + 2U/c^2) \delta_{ij}$. Therefore, the elapsed TCG is:

$$\Delta t \approx \frac{1}{c} \int_{\text{path}} \sqrt{\frac{g_{ij}}{-g_{00}}} dx^i dx^j \approx \frac{1}{c} \int_{\text{path}} \left(1 + \frac{2}{c^2} U\right) \sqrt{\delta_{ij} dx^i dx^j} = \frac{\rho}{c} + \frac{1}{c^3} \int_{\text{path}} 2U d\rho \quad (37)$$

The gravitational time delay is:

$$\Delta t_{\text{delay}} = \frac{2GM}{c^3} \ln \left(\frac{R+r+\rho}{R+r-\rho} \right) \quad (38)$$

where:

R and r : the distances from the geocenter to the transmitter and receiver, respectively

The gravitational delay typically amounts to a few tens of picoseconds for a path between a satellite and Earth. The total TCG is the sum of the terms in equations (35) and (38).

The coordinate time of propagation (TT) is:

$$\Delta t' = (1 - L_G) \Delta t = \frac{\rho}{c} - L_G \frac{\rho}{c} + \frac{2GM}{c^3} \ln \left(\frac{R+r+\rho}{R+r-\rho} \right) \quad (39)$$

This is the time interval that would be measured by a clock on the geoid.

For example, for a signal sent from a geostationary satellite with orbital radius 42 164 km to a clock on the equator at the same longitude, the path delay is -27 ps. For a GPS satellite at an angle of elevation of 40° , the second and third terms nearly cancel and the path delay is -3 ps.

Earth-Centered Earth-Fixed coordinate system

When considering the computation in an ECEF coordinate system, the geometric part of the TCG is:

$$\Delta t = \frac{1}{c} \int_{\text{path}} \sqrt{g_{ij} dx^i dx^j} + \frac{1}{c} \int_{\text{path}} g_{0j} dx^j \quad (40)$$

The metric components are $-g_{00} \approx 1$, $g_{0j} = (\boldsymbol{\omega} \times \mathbf{r})_j / c$, and $g_{ij} \approx \delta_{ij}$, where \mathbf{r} is the position vector of a point on the signal path. The coordinate time (TT) is $\Delta t' = (1 - L_G) \Delta t$.

The first term of equation (40) is ρ' / c , where ρ' is the Euclidean path length in the ECEF coordinate system. If the transmitter has position \mathbf{r}_T and the receiver has position \mathbf{r}_R and velocity \mathbf{v}'_R , then

$$\frac{\rho'}{c} = \frac{1}{c} |\mathbf{r}_R(t_R) - \mathbf{r}_T(t_T)| \approx \frac{1}{c} |\Delta \mathbf{r} + \mathbf{v}'_R(t_R - t_T)| \approx \frac{1}{c} |\Delta \mathbf{r}| + \frac{1}{c^2} \Delta \mathbf{r} \cdot \mathbf{v}'_R \quad (41)$$

where:

$$\Delta \mathbf{r} \equiv \mathbf{r}_R(t_T) - \mathbf{r}_T(t_T)$$

The second term of equation (40) is the Sagnac effect. Therefore,

$$\Delta t_{\text{Sagnac}} \approx \frac{1}{c^2} \int_A^B (\boldsymbol{\omega} \times \mathbf{r}) \cdot d\mathbf{r} = \frac{1}{c^2} \int_A^B \boldsymbol{\omega} \cdot (\mathbf{r} \times d\mathbf{r}) = 2 \frac{1}{c^2} \int_A^B \boldsymbol{\omega} \cdot d\mathbf{A} = \frac{2\boldsymbol{\omega} \cdot \mathbf{A}}{c^2} \quad (42)$$

where:

- A : the projection onto the equatorial plane of the area formed by the center of rotation and the endpoints of the signal path.

The gravitational delay must be considered as well to compute the total time of propagation.

Barycentric coordinate system

To describe the propagation of an electromagnetic signal, a barycentric coordinate system with Cartesian coordinates (x, y, z) may be used.

Because only the gravitational effect of the Sun is considered here, for convenience of computation of the gravitational delay, one may use a space grid where the transmitter position is $(-a_T, b, 0)$ and the receiver position is $(a_R, b, 0)$ so that the propagation is along the approximately straight line path $y = b$ (neglecting gravitational deflection), where b is the distance of closest approach to the Sun. The coordinate time of propagation (TCB) is:

$$\Delta t \approx \frac{1}{c} \int_{\text{path}} \sqrt{\frac{g_{ij}}{-g_{00}}} dx^i dx^j \approx \frac{1}{c} \int_{-a_T}^{a_R} \left(1 + \frac{2}{c^2} U_S \right) dx = \frac{1}{c} \int_{-a_T}^{a_R} \left(1 + \frac{1}{c^2} \frac{2GM_S}{\sqrt{x^2 + b^2}} \right) dx \quad (43)$$

where U_S is the gravitational potential of the Sun. Therefore,

$$\Delta t = \frac{1}{c}(a_T + a_R) + 2 \frac{GM_S}{c^3} \ln \frac{a_R + \sqrt{a_R^2 + b^2}}{-a_T + \sqrt{a_T^2 + b^2}} \quad (44)$$

To a level of approximation depending on the propagation time, the TT coordinate time of propagation may be scaled from the TCB time as:

$$\Delta t' = (1 - L_B) \Delta t = \frac{1}{c}(1 - L_B)(a_T + a_R) + 2 \frac{GM_S}{c^3} \ln \frac{a_R + \sqrt{a_R^2 + b^2}}{-a_T + \sqrt{a_T^2 + b^2}} \quad (45)$$

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