RECOMMENDATION ITU-R TF.1010-1*

Relativistic effects in a coordinate time system in the vicinity of the Earth

(Question ITU-R 152/7)

(1994-1997)

The ITU Radiocommunication Assembly,

considering

a) that it is desirable to maintain coordination of standard frequency and time-signal emissions in the vicinity of the Earth;

b) that universal coordinated time (UTC) is the official coordinate time scale for the Earth defined on the rotating geoid;

c) that atomic clocks are subject to path-dependent second-order motional frequency shifts and position-dependent gravitational frequency shifts;

d) that the Consultative Committee for Time and Frequency (CCTF, formerly CCDS) has recognized the need for well-defined procedures to account for relativistic effects in timing systems and time comparisons;

e) that, since time comparisons in non-inertial frames require special consideration and the CCDS has recommended an appropriate set of equations which provide a consistent set of measurements of UTC in the vicinity of the Earth;

f) that there is a growing trend to place accurate, stable clocks in Earth-bound orbits for time-keeping purposes;

g) that there is a need for comparing frequency standards in the vicinity of the Earth with an accuracy of 10^{-14},

recommends

1 that for calculating coordinate time intervals in the vicinity of the Earth (out to at least geosynchronous radius) to an accuracy of 1 ns (or 10^{-14} of the integration time), the following procedures, based on the first order terms in the full general relativistic expressions, should be followed (some practical examples are given in Annex 1):

1.1 Clock transport in a rotating reference frame

When transferring time from point P to point Q by means of a portable clock, the coordinate time accumulated during transport is:

$$\Delta t = \int_P^Q ds \left[1 + \frac{\Delta U(\vec{r})}{c^2} + \frac{V^2}{2c^2} \right] + \frac{2\,\omega}{c^2} A_E \qquad (1)$$

* Radiocommunication Study Group 7 made editorial amendments to this Recommendation in 2003 in accordance with Resolution ITU-R 44.

where:

c: speed of light

ω: angular velocity of rotation of the Earth

V: velocity of the clock with respect to the ground

$\vec{r}$: vector whose origin is at the centre of the Earth and whose terminus moves with the clock from P to Q

A_E: equatorial projection of the area swept out during the time transfer by the vector $\vec{r}$ as its terminus moves from P to Q

$\Delta U(\vec{r})$: gravitational potential difference (including the centrifugal potential) between the location of the clock at $\vec{r}$ and the geoid as viewed from an earth-fixed coordinate system, in agreement with the convention (Resolution A4, IAU, 1992) that $\Delta U(\vec{r})$ is negative when the clock is above the geoid

ds: increment of proper time accumulated on the portable clock. The increment of proper time is the time accumulated on the portable standard clock as measured in the "rest frame" of the clock; that is, in the reference frame travelling with the clock.

A_E is measured in an earth-fixed coordinate system. As the area A_E is swept, it is taken as positive when the projection of the path of the clock on the equatorial plane moves eastward. When the height h of the clock is less than 24 km above the geoid, $\Delta U(\vec{r})$ may be approximated by gh, where g is the total acceleration due to gravity (including the rotational acceleration of the Earth) evaluated at the geoid. This approximation applies to all aerodynamic and earthbound transfers.

When h is greater than 24 km, the potential difference $\Delta U(\vec{r})$ must be calculated to greater accuracy as follows:

$$\Delta U(\vec{r}) = GM_e/r + J_2 \; GM_e \, a_1^2 \, (1 - 3\cos^2\theta)/2r^3 + \omega^2 \, r^2 \, \sin^2\theta/2 - U_g \qquad (2)$$

where:

a_1: equatorial radius of the Earth

$a_1 = 6\,378.136$ km

r: magnitude of the vector $\vec{r}$

θ: colatitude

GM_e: product of the Earth's mass and the gravitational constant

$GM_e = 398\,600$ km^3/s^2

J_2: quadrupole moment coefficient of the Earth

$J_2 = +1.083 \times 10^{-3}$

ω: angular velocity of the Earth

$\omega = 7.292115 \times 10^{-5}$ rad/s

U_g: potential (gravitational and centrifugal) at the geoid

$U_g = 62.63686$ km^2/s^2.

For time transfer at the 1 ns level of accuracy, this expression should not be used beyond a distance of about 50 000 km from the centre of the Earth.

1.2 Clock transport in a non-rotating reference frame

When transferring time from point P to point Q by means of a clock the coordinate time elapsed during the motion of the clock is:

$$\Delta t = \int_P^Q ds \left[1 + \frac{U(\vec{r}) - Ug}{c^2} + \frac{v^2}{2c^2} \right] \tag{3}$$

where:

- $U(\vec{r})$: gravitational potential at the location of the clock excluding the centrifugal potential
- v: velocity of the clock, both as viewed (in contrast to equation (1)) from the geocentric non-rotating reference frame
- U_g: potential at the geoid

  ($U_g/c^2 = -6.9694 \times 10^{-10}$), including the effect on the potential of the Earth's rotational motion.

Note that $\Delta U(\vec{r}) \neq U(\vec{r}) - U_g$, since $U(\vec{r})$does not include the effect of the Earth's rotation. This equation also applies to clocks in geostationary orbits but should not be used beyond a distance of about 50 000 km from the centre of the Earth.

1.3 Electromagnetic signals in a rotating reference frame

From the viewpoint of a geocentric, earth-fixed, rotating frame, the coordinate time elapsed between emission and reception of an electromagnetic signal is:

$$\Delta t = \frac{1}{c} \int_P^Q d\sigma \left[1 + \frac{\Delta U(\vec{r})}{c^2} \right] + \frac{2\omega}{c^2} A_E \tag{4}$$

where:

- $d\sigma$: increment of standard length, or proper length, along the transmission path
- $\Delta U(\vec{r})$: potential at the point, $\vec{r}$, on the transmission path less the potential at the geoid (see equation (3)), as viewed from an earth-fixed coordinate system
- A_E: area circumscribed by the equatorial projection of the triangle whose vertices are:
  - at the centre of the Earth
  - at the point, P, of transmission of the signal
  - at the point, Q, of reception of the signal.

The area, A_E, is positive when the signal path has an eastward component. The second term amounts to about one tenth nanosecond for an Earth-to-geostationary satellite-to-Earth trajectory. In the third term, $2\omega/c^2 = 1.6227 \times 10^{-6}$ ns/km^2; this term can contribute hundreds of nanoseconds for practical values of A_E. The increment of proper length, $d\sigma$, can be taken as the length measured using standard rigid rods at rest in the rotating system; this is equivalent to measurement of length by taking $c/2$ times the time (normalized to vacuum) of a two-way electromagnetic signal sent from P to Q and back along the transmission path.

1.4 Electromagnetic signals in a non-rotating reference frame

From the viewpoint of a geocentric non-rotating (local inertial) frame, the coordinate time elapsed between emission and reception of an electromagnetic signal is:

$$\Delta t = \frac{1}{c} \int_P^Q d\sigma \left[1 + \frac{U(\vec{r}) - Ug}{c^2} \right] \tag{5}$$

where $U(\vec{r})$ and U_g are defined as in equation (3), and $d\sigma$ is the increment of standard length, or proper length, along the transmission path.

Annex 1

Examples

Due to relativistic effects, a clock at an elevated location will appear to be higher in frequency and will differ in normalized rate from International Atomic Time (TAI) by:

$$-\frac{\Delta U}{c^2}$$

where:

ΔU: difference in the total potential (gravitational and the centrifugal potentials)

c: velocity of light.

Near sea level this is given by:

$$-\frac{g(\varphi)\, h}{c^2} \tag{6}$$

where:

φ: geographical latitude

$g(\varphi)$: total acceleration at sea level (gravitational and centrifugal)

$g(\varphi) = (9.780 + 0.052 \sin^2 \varphi)$ m/s^2

h: distance above sea level.

Equation (6) must be used in comparing primary sources of the SI second with TAI and with each other. For example, at latitude 40°, the rate of a clock will change by $+1.091 \times 10^{-13}$ for each kilometre above the rotating geoid.

If a clock is moving relative to the Earth's surface with speed V, which may have a component V_E in the direction to the East, the normalized difference in frequency of the moving clock relative to that of a clock at rest at sea level is:

$$-\frac{1}{2} \frac{V^2}{c^2} + \frac{g(\varphi)\, h}{c^2} - \frac{1}{c^2} \cdot \omega \cdot r \cdot \cos\varphi \cdot V_E \tag{7}$$

where:

ω: angular rotational velocity of the Earth

$\omega = 7.292 \times 10^{-5}$ rad/s

r: distance of the clock from the centre of the Earth (equatorial Earth radius = 6 378.136 km)

c: velocity of light

$c = 2.99792458 \times 10^{5}$ km/s

φ: geographical latitude.

For example, if the clock is moving 270 m/s East at 40° latitude at an altitude of 9 km, the normalized difference in frequency of the moving clock relative to that of a clock at rest at sea level due to this effect is:

$$-4.06 \times 10^{-13} + 9.82 \times 10^{-13} - 1.072 \times 10^{-12} = -4.96 \times 10^{-13}$$

The choice of a coordinate frame is purely a discretionary one, but to define coordinate time, a specific choice must be made. It is recommended that for terrestrial use an earth-fixed frame be chosen. In this frame, when a clock B is synchronized with a clock A (both clocks being stationary on the Earth) by a radio signal travelling from A to B, these two clocks differ in coordinate time by:

$$t_B - t_A = -\frac{\omega}{c^2} \int_p d\lambda \; r^2 \cos^2 \varphi \tag{8}$$

where:

φ: latitude

λ: longitude (the positive sense being toward East)

p: path over which the radio signal travels from A to B.

If the two clocks are synchronized by a portable clock, they will differ in coordinate time by:

$$t_B - t_A \int_p ds \left(\frac{\Delta U(\vec{r})}{c^2} - \frac{V^2}{2c^2} \right) - \frac{\omega}{c^2} \int_p d\lambda \; r^2 \cos^2 \varphi \tag{9}$$

where:

V: portable clock's ground speed

p: portable clock's path from A to B.

This difference can also be as much as several tenths of a microsecond. It is recommended that equations (8) or (9) be used as correction equations for long-distance clock synchronization. Since equations (8) and (9) are path dependent, they must be taken into account in any self-consistent coordinate time system.

If a clock is transported from a point A to a point B and brought back to A on a different path at infinitely low speed at $h = 0$, its time will differ from that of a clock remaining at A by:

$$\Delta t = \frac{2\omega A_E}{c^2}$$

where A_E is the area defined by the projection of the round trip path on to the plane of the Earth's equator. A_E is considered positive if the path is traversed in the clockwise sense viewed from the South Pole.

For example since:

$$2\omega / c^2 = 1{,}6227 \times 10^{-6} \text{ ns/km}^2$$

the time of a clock carried eastward around the Earth at infinitely low speed at $h = 0$ at the equator will differ from a clock remaining at rest by –207.4 ns.

At the 10^{-14} level of correction, heights above sea level, above the rotating geoid and the Global Positioning System (GPS) indicated height are all equivalent.