MINIMUM COLLISIONS ASSIGNMENT IN INTERDEPENDENT NETWORKED SYSTEMS VIA DEFECTIVE COLORINGS

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Abstract – *In conjunction with the traffic overload of next-generation wireless communication and computer networks, their resource-constrained nature calls for effective methods to deal with the fundamental resource allocation problem. In this context, the Minimum Collisions Assignment (MCA) problem in an interdependent networked system refers to the assignment of a finite set of resources over the nodes of the network, such that the number of collisions, i.e., the number of interdependent nodes receiving the same resource, is minimized. Given the interdependent networked system's organization in the form of a graph, there already exists a randomized algorithm that converges with high probability to an assignment of resources having zero collisions when the number of resources is larger than the maximum degree of the underlying graph. In this article, differing from the prevailing literature, we investigate the case of a resource-constrained networked system, where the number of resources is less than or equal to the maximum degree of the underlying graph. We introduce two distributed, randomized algorithms that converge in a logarithmic number of rounds to an assignment of resources over the network for which every node has at most a certain number of collisions. The proposed algorithms apply to settings where the available resources at each node are equal to three and two, respectively, while they are executed in a fully-distributed manner without requiring information exchange between the networked nodes.*

Keywords – Defective coloring, games on graphs, graph coloring, resource allocation

1. INTRODUCTION

Next-generation communication and computer networks comprise an ecosystem of interconnected entities where multiple stakeholders and devices evolve in a physical, digital, or virtual space with others, resulting in interdependent interactions between them. Particular realizations of such systems include 6G subnetworks, Internet of Things (IoT) and sensor networks, Multi-access Edge Computing (MEC) and caching networks, and the intersection of these technologies with other disciplines, e.g., smart cities, industries [1, 2]. Although being part of a broader network infrastructure, emerging networked systems should be able to orchestrate and configure autonomously due to scalability, resilience, and security reasons, practically implying a loose interrelation between them regarding the information exchange. At the same time, though, these systems present competitive environments at their basis where the actions are not only interdependent but also conflicting. Therefore, effective methods to deal with the fundamental problem of resource allocation in a distributed manner while relying solely on local information are required.

In this article, following a graph-based representation and reasoning, we consider such an interdependent networked system and study the constrained resource allocation problem with resource conflicts. The considered system is represented in the form of a finite graph, whose vertices and edges correspond to the networked nodes and their in-between dependencies, respectively, and which will be referred to as the *underlying graph* of the system. In this context, we are concerned with the problem of assigning a finite set of resources to the nodes of the network in such a way that the number of *collisions*, i.e., the number of pairs of interdependent nodes that are assigned the same resource, is minimized. The respective optimization problem is known as the Minimum Collisions Assignment (MCA) problem. Targeting a solution that is applicable in a wide array of contexts, we equivalently consider the problem of assigning colors to the vertices of the underlying graph instead of assigning resources to the nodes of the network. Then, the MCA problem is translated as the problem of assigning a given number of colors over the vertices of the graph in such a way that the number of monochromatic edges is minimal. The aforementioned problem is known to be NP-hard (see [3]).

Generally, several applications of the classical Graph Coloring Problem (GCP) exist in interdependent networked systems, while the most well-known is channel allocation and sharing in wireless networks [4, 5, 6, 7, 8, 9]. The problem is translated into coloring an interference graph where the colors are the channels. Distributed greedy and randomized algorithms that target interference mitigation and throughput maximization have been introduced therein [4]. Building upon these algorithms, channel allocation solutions to more specialized network settings, e.g., Wireless Body Area Networks (WBANs) [5], 6G subnetworks [6], and cognitive radio networks [7], can also be found. Going one step further in the complexity of the considered resource allocation problem regarding the number of resources to be allocated, more recent work in [8, 9] study the joint Resource Block (RB) allocation and power control in Full Duplex (FD) wireless networks using centralized greedy dual graph coloring algorithms. Other applications of graph coloring algorithms regard content placement and caching in modern small-cell networks [9, 10]. The vertices of the underlying graph correspond to the base stations where the files are cached, and, eventually, the colors represent the files. The goal is to optimize the content retrieval by the users and reduce the transmission time through the backhaul.

Nevertheless, the overwhelming majority of literature so far (e.g., [4, 5, 6, 7, 8, 9]) has focused on MCA problems for which the number of available colors (resources) is larger than the maximum degree of the underlying graph, and several algorithms have been proposed therein, both centralized and distributed, that result in colorings of zero collisions. It appears that instances of the MCA problem with "few", i.e., less than or equal to the maximum degree of the underlying graph, colors are much less investigated and, in this article, we aim to exactly fill this gap. Respecting the need for distributed resource allocation, we capitalize on each networked node's local awareness concerning its neighbors' allocated resources and propose a distributed approach to the MCA problem in the case of there being "few" available colors. In more detail, our main result builds upon a game-theoretic model for defective graph coloring, for which we propose the Greedy and Frugal, symmetric, randomized strategies for the networked nodes. This in turn results in two distributed randomized algorithms for the MCA problem. In the following, we refer to the entities/nodes of an interdependent networked system as players, and the set of resources as colors.

The remainder of this article is organized as follows. Section 2 provides the basic definitions along with the problem formulation, which is stated in graph-theoretic terminology to attain generality. Section 3 includes a review of the literature relevant to the problem. In Section 4, we state our main results and findings, along with the designed game-theoretic strategies and their pseudocodes. The detailed mathematical analysis and proof of our main result are included in Section 5. In Section 6, numerical results obtained via modeling and simulation are presented to support our theoretical analysis, and Section 7 concludes the paper.

2. BASIC DEFINITIONS AND PROBLEM FORMULATION

All graphs considered in this article are finite, without loops, and undirected. Throughout the article, given a positive integer k, we denote by $[k]$ the set $\{1,\dots,k\}$ and, given a finite set F, we denote by $|F|$ its cardinality. Given a graph $G=(V,E)$ and a vertex $v\in V$, we let $\mathcal{N}_G(v)=\{u\in V : (u,v)\in E\}$ be the *neighborhood* of v in G. The *degree* of v equals $\deg_G(v)=|\mathcal{N}_G(v)|$ and the maximum degree of vertices in G is denoted Δ_G. A k*-coloring* of G is a function $\chi : V\to[k]$. Given a k-coloring χ of a graph $G=(V,E)$, and a subset $A\subset V$, we denote by $\chi(A):=\bigcup_{v\in A}\{\chi(v)\}$ the set of colors of the vertices in A. Moreover, the *collision number* of χ is defined as $\mathcal{C}_G(\chi)=|\{e=(u,v)\in E : \chi(u)=\chi(v)\}|$, and the *collision number of a vertex* $v\in V$ under a k-coloring χ of G is defined as $\mathcal{C}_G(v;\chi)=|\{u\in\mathcal{N}_G(v) : \chi(u)=\chi(v)\}|$. In other words, $\mathcal{C}_G(\chi)$ equals the cardinality of the set consisting of all *monochromatic edges* of G under the coloring χ, and $\mathcal{C}_G(v;\chi)$ is the number of neighbors of v that receive the same color as v.

A k-coloring χ of $G=(V,E)$ is called s*-colliding* if $\mathcal{C}_G(\chi)\le s$; it is called d*-defective* if $\mathcal{C}_G(v;\chi)\le d$ holds true for all $v\in V$. A 0-colliding coloring is referred to as a *proper coloring* in the literature. In other words, the MCA problem is a graph coloring problem which is equivalent to the problem of determining $\mathcal{C}_k(G):=\min_\chi \mathcal{C}_G(\chi)$, where the minimum is over all k-colorings of G.

3. MOTIVATION AND CONTRIBUTIONS

By exploring the related literature, it appears that the classical GCP algorithms find a wide application in resource allocation problems in wireless networks. From a theoretical viewpoint, in the classical GCP, the objective is to find the minimum positive integer k for which a given graph G on n vertices admits a 0-colliding k-coloring. This minimum value of k is referred to as the *chromatic number* of G, and is denoted $chr(G)$. The MCA problem is, in some sense, dual to the GCP. In the setting of the MCA problem, the parameter k is fixed and the objective is to find a k-coloring of a given graph G on n vertices that has a minimum collision number, among all k-colorings of G. Observe that when $k\ge chr(G)$ then the GCP implies that the MCA problem admits a 0-colliding k-coloring. In particular, it is well known that when $k\ge\Delta_G+1$ the graph G admits a 0-colliding k-coloring, and one can obtain such a coloring using a variety of algorithms, both centralized and distributed. It is also well known that when $k\ge\Delta_G+1$, a 0-colliding coloring of G can be found in linear time by a centralized algorithm. However, the problem becomes more delicate when the algorithm is required to be distributed. In this work, we shall be interested in distributed algorithms for the MCA problem.

Distributed algorithms for GCP are extensively analyzed and discussed in [11], focusing on both deterministic and randomized instances. A representative example of a distributed deterministic algorithm for the GCP is presented in [12], managing to conclude a graph coloring in linear $O(\Delta_G)$ time. In particular, the work in [12] belongs to the broader category of graph coloring algorithms that utilize an initial defective coloring to conclude with zero collisions. Concerning the distributed randomized algorithms, both works in [13] and [14] achieve to determine a 0-colliding coloring of a graph in $O(\log(n))$ rounds when the number of available colors is at least $\Delta + 1$. However, they are limited in that they require each player to communicate with their neighbors whether they have any conflict or not. The state of the art behind distributed randomized algorithms can be found in [15], where an algorithmic complexity of $O(\log^3(\log(n)))$ rounds is achieved. Considering the algorithm instances that do not require any communication between the players, preliminary work is provided in [16]. The respective algorithm yields a 0-colliding coloring of a graph in $O(\log(n))$ rounds when the number of colors available is at least $\Delta_G + 2$. The algorithm from [16] has been further improved in [3] to a distributed instance that yields a 0-colliding coloring of a graph in at most $O(\Delta_G \cdot \log(n))$ rounds when the number of available colors is at least $\Delta_G + 1$, with the cost of requiring communication among neighbors. Another improvement of the algorithm in [16], which assumes no cooperation/communication among neighbors and which yields a 0-colliding coloring in $O(\log(n))$ rounds when the number of available colors is at least $\Delta_G + 1$, can be found in [17]. In other words, when $k \geq \Delta_G + 1$, it holds $\mathcal{C}_G(k) = 0$, for any graph G.

In this article, we focus on instances of the MCA problem for which $k \leq \Delta_G$ which, to the best of our knowledge, appear to be less investigated. One approach to the problem is to allow the possibility of leaving some vertices uncolored, and thus employ *incomplete* 0-colliding k-colorings of the underlying graph (see [18] and references therein). Another approach is based on *dispersion games* (see [19] and [20]), but only applies to instances of the MCA problem for which the underlying graph is complete. Our approach is based on defective colorings (see [21]), and builds upon ideas from [16]. In particular, in [16], the authors define the *network coloring game*, which is played on a graph G, and study the dynamics of the game when the players adopt a particular greedy, randomized, strategy. It is shown, in [16], that the dynamics of the network coloring game under the aforementioned greedy strategy converge to a Nash equilibrium that gives rise to a 0-colliding k-coloring of G, provided that $k \geq \Delta_G + 2$.

In a conference version of this article (see [22]), we introduced the *defective coloring game* and studied its dynamics when a particular greedy, randomized, strategy is adopted by the players. We demonstrated that the dynamics of this greedy strategy converge to a Nash equilibrium which also provides a defective coloring of the underlying graph. In this article, we improve upon the aforementioned greedy strategy. Our improvement is two-fold. On the one hand, we provide an improved version of the greedy strategy, which is referred to as *frugal strategy*. The frugal strategy allows for a reduction of the number of available colors to each player by one, thus applying to settings in which the greedy strategy does not apply. In particular, the frugal strategy applies when the number of available colors to each player is equal to 2. On the other hand, we improve slightly on the upper bound on the number of collisions, i.e., Corollary 1, from [22].

4. MAIN RESULT: DEFECTIVE COLORING GAME

In this section, we define the *defective coloring game*, which has been introduced in [22] and may be seen as a variant of the *network coloring game* from [16]. Fix a graph $G = (V, E)$ having $n = |V|$ vertices and maximum degree Δ_G, as well as two integers k, d such that $k \in \{2, \dots, \Delta_G\}$ and $d \in [\Delta_G - 1]$. The defective coloring game on the graph G, denoted $DCG(G; k, d)$, is defined as follows.

The players of $DCG(G; k, d)$ are the vertices of G and participate in a game that is played over a number of rounds. In every round, all players simultaneously and individually choose a color from their set of available colors, which is assumed to be the set $[k]$. Thus, after round t, the choices of the players give rise to a k-coloring of G, which is denoted χ_t. The players of $DCG(G; k, d)$ have only local information on the graph: they can only observe the colors chosen by their neighbors and are not allowed to communicate or cooperate. A player $v \in V$ is said to be *happy* after round t if their collision number under χ_t is at most d; i.e., when $\mathcal{C}_G(v; \chi_t) \leq d$. Otherwise, the player is *unhappy*. The payoff to a player in $DCG(G; k, d)$ is 1 when they are happy, and 0 when they are unhappy, and a configuration of colors for which every player receives payoff 1 is a *Nash equilibrium* of the $DCG(G; k, d)$ in that no player has the incentive to unilaterally change strategy under such a configuration. Observe that, when the players have chosen colors that constitute a Nash equilibrium, the corresponding k-coloring of the graph is d-defective.

The challenge is to find a *symmetric strategy*, i.e., a common strategy for all players in $DCG(G; k, d)$, that achieves converging to a Nash equilibrium after a finite number of rounds using the available colors. Such a strategy has been proposed in [22], and is referred to as the *Greedy strategy*. Let $\chi_t(v)$ denote the color chosen by player $v \in V$ after round t, and let $\chi_t(\mathcal{N}(v))$ be the set of colors chosen by the neighbors of v after round t. Then, the greedy strategy is summarized as follows.

Greedy strategy. Suppose that $k = \Delta_G - s$ and $d = s + 2$, for some fixed $s \in \{0, 1, \dots, \Delta_G - 3\}$. Assume further that each player in $DCG(G; k, d)$ adopts the following strategy: if a player v is happy after a certain round t, then they stick to their choice in all subsequent rounds, i.e., $\chi_s(v) = \chi_t(v)$, for all $s > t$. If they are unhappy then in the next round they change color, and choose uniformly at random a color from the set $[k] \ \ \chi_t(\mathcal{N}(v))$.

In other words, under the greedy strategy, a player who is unhappy after a certain round t chooses in the next round a color uniformly at random from the set consisting of those colors that are *not* chosen by their neighbors after round t. The corresponding algorithm is summarized in Algorithm 1.

Remark 1. *Notice that, when the players of $DCG(G; k, d)$, with $k = \Delta_G - s$ and $d = s + 2$, adopt the greedy strategy, a player who is happy after a certain round remains happy in all subsequent rounds. Furthermore, if player v is unhappy after round t, then it holds $|[k] \ \ \chi_t(\mathcal{N}(v))| \geq 2$. In particular, an unhappy player has always at least two available colors to choose from in the next round.*

Now, suppose that all players in $DCG(G; k, d)$ adopt the greedy strategy. The main result from [22] reads as follows.

Theorem 1. *Let G be a graph on n vertices and maximum degree $\Delta_G \geq 3$. Let k, d be fixed positive integers such that $k = \Delta_G - s$ and $d = s + 2$, for some $s \in \{0, 1, \dots, \Delta_G - 3\}$. Suppose that each player in $DCG(G; k, d)$ adopts the greedy strategy. Then, for any starting assignment of colors to the vertices, the defective coloring game on G converges after at most $O(\log(\frac{n}{\delta}))$ rounds with a probability of at least $1 - \delta$.*

In other words, when s does not depend on n and the players in the defective coloring game adopt the greedy strategy, the game reaches a d-defective k-coloring of the graph in $O(\log(\frac{n}{\delta}))$ rounds with a probability of at least $1 - \delta$.

Corollary 1. *Let G be a graph on $n = |V|$ vertices and maximum degree $\Delta_G \geq 3$. Suppose that $k = \Delta_G - s$, for some $s \in \{0, 1, \dots, \Delta_G - 3\}$. Then, it holds $\mathcal{C}_k(G) \leq \frac{n(s+2)}{2}$.*

The proofs of Theorem 1 and Corollary 1 are analytically presented in [22].

In this article, we show that a modification of the above-mentioned greedy strategy provides an improved version of Theorem 1. The "modified greedy strategy" is referred to as a *Frugal strategy*, and is formally defined as follows.

Frugal strategy. Suppose that $k = \Delta_G - s$ and $d = s + 1$, for some fixed $s \in \{0, 1, \dots, \Delta_G - 2\}$. Assume further that each player in $DCG(G; k, d)$ adopts the following strategy: if a player, say v, is happy after a certain round, say t, then they stick to their choice in all subsequent rounds, i.e., $\chi_s(v) = \chi_t(v)$, for all $s > t$. If they are unhappy then in the next round they change color, and choose uniformly at random a color from the set $\{\chi_t(v)\} \cup ([k] \ \ \chi_t(\mathcal{N}(v)))$.

In other words, under the frugal strategy, a player who is unhappy after round $t \geq 1$ chooses in the next round a color uniformly at random from the set consisting of their color choice after round t and the set of colors that are not chosen by their neighbors after round t. The corresponding algorithm is summarized in Algorithm 2.

Algorithm 1 Greedy strategy algorithm

Input $G = (V, E)$, Δ_G, $s \in \{0, 1, \dots, \Delta_G - 3\}$
Output $\chi_t(v), \forall v \in V$

$d \leftarrow s + 2$
$k \leftarrow \Delta_G - s$
for each $v \in V$ **do**
  Choose $\chi_1(v)$ randomly from the set $[k]$
  $t \leftarrow 1$
  while $\mathcal{C}_G(v; \chi_t) \geq d + 1$ **do**
    $\mathcal{A}_t(v) \leftarrow [k] \ \ \chi_t(\mathcal{N}(v))$
    Choose a color $\chi_{t+1}(v)$ uniformly at random from the set $\mathcal{A}_t(v)$
    $t \leftarrow t + 1$
  end while
  Return $\chi_t(v)$
end for

Algorithm 2 Frugal strategy algorithm

Input $G = (V, E)$, Δ_G, $s \in \{0, 1, \dots, \Delta_G - 2\}$
Output $\chi_t(v), \forall v \in V$

$d \leftarrow s + 1$
$k \leftarrow \Delta_G - s$
for each $v \in V$ **do**
  Choose $\chi_1(v)$ randomly from the set $[k]$
  $t \leftarrow 1$
  while $\mathcal{C}_G(v; \chi_t) \geq d + 1$ **do**
    $\mathcal{A}_t(v) \leftarrow [k] \ \ \chi_t(\mathcal{N}(v))$
    $\mathcal{B}_t(v) \leftarrow \{\chi_t(v)\} \cup \mathcal{A}_t(v)$
    Choose a color $\chi_{t+1}(v)$ uniformly at random from the set $\mathcal{A}_t(v)$
    $t \leftarrow t + 1$
  end while
  Return $\chi_t(v)$
end for

Remark 2. *Notice that, when the players of $DCG(G; k, d)$, with $k = \Delta_G - s$ and $d = s + 1$, adopt the frugal strategy, a player who is happy after a certain round remains happy in all subsequent rounds. Furthermore, if player v is unhappy after round t, then it holds $|\{\chi_t(v)\} \cup ([k] \ \ \chi_t(\mathcal{N}(v)))| \geq 2$. In particular, an unhappy player has always at least two available colors to choose from in the next round. Notice also that, in contrast to the greedy strategy, under the frugal strategy, an unhappy player* may not change color *in the next round. Finally, observe that the frugal strategy works when $k \geq 2$, in contrast to the greedy strategy which requires at least three available colors.*

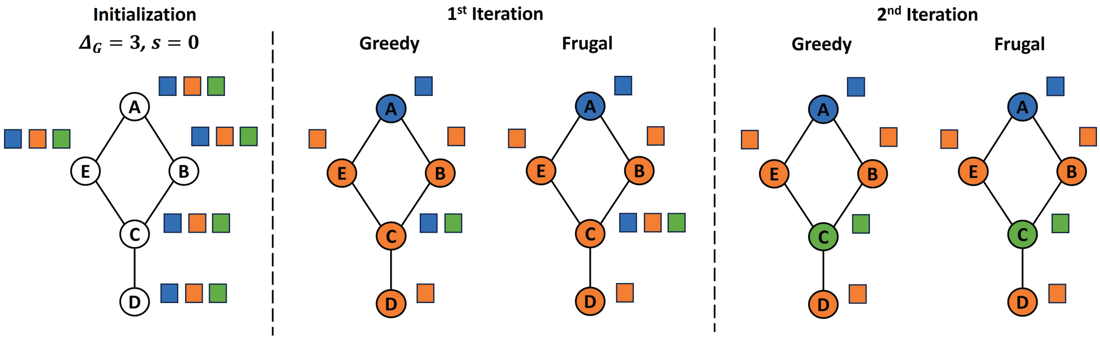

Figure 1 – Overview of the operation of the greedy and frugal strategies under a graph of maximum degree $\Delta_G = 3$ for $s = 0$.

Our main result implies that the conclusion of Theorem 1 remains unchanged when the players adopt the frugal strategy.

Theorem 2. *Let G be a graph on n vertices and maximum degree $\Delta_G \geq 2$. Let k, d be fixed positive integers such that $k = \Delta_G - s$ and $d = s+1$, for some $s \in \{0, 1, \dots, \Delta_G - 2\}$. Suppose that each player in $DCG(G; k, d)$ adopts the frugal strategy. Then, for any starting assignment of colors to the vertices, the defective coloring game on G converges after at most $O(\log(\frac{n}{\delta}))$ rounds with a probability of at least $1 - \delta$.*

In other words, when s does not depend on n and the players in the defective coloring game adopt the frugal strategy, the game reaches a d-defective k-coloring of the graph in $O(\log(\frac{n}{\delta}))$ rounds with a probability of at least $1-\delta$. Now, the number of monochromatic edges in such a coloring provides an upper bound on the quantity $\mathcal{C}_G(k)$, in the MCA problem. In particular, Theorem 2 yields the following.

Corollary 2. *Let G be a graph on $n = |V|$ vertices and maximum degree $\Delta_G \geq 2$. Suppose that $k = \Delta_G - s$, for some $s \in \{0, 1, \dots, \Delta_G - 2\}$. Then, it holds $\mathcal{C}_k(G) \leq \frac{n(s+1)}{2}$.*

Proof. Consider the defective-coloring game $DCG(G; k, d)$, where $d = s + 1$. From Theorem 2 we know that, when the players in $DCG(G; k, d)$ adopt the frugal strategy, the game reaches a Nash equilibrium in $O(\log(n))$ expected number of rounds. Let χ be the k-coloring corresponding to a Nash equilibrium. In such an equilibrium point, all vertices are happy and the collision number of each vertex $v \in V$ is at most d. Let G_χ be the graph induced by the monochromatic edges of G under χ. Then, $\mathcal{C}_G(\chi)$ equals the number of edges in G_χ, and the result follows from the degree-sum formula in G_χ. □

To visualize the operation and basic steps of the proposed greedy and frugal strategies, an illustrative example is presented in Fig. 1 for a graph with maximum degree $\Delta_G = 3$. In the left part of Fig. 1, the initialization of the graph is depicted, where it is assumed that $s = 0$. Therefore, the number of available colors of each vertex is $k = 3$, namely blue, orange, and green, while $d = 2$ and $d = 1$ for the greedy and frugal strategies, respectively. In the first iteration of both strategies, the players, i.e., nodes, randomly select a color from their palette as shown. The selected colors render all nodes happy except for node C. The nodes stick to their color choice, contrary to node C which updates its color palette with the available colors. At this point, the difference in the updated color palette between the two strategies is visualized. In the second iteration of both strategies, algorithmic convergence is met.

5. PROOF OF THEOREM 2

In this section, we prove Theorem 2. We fix a starting assignment of colors to the vertices, and we assume that each player in $DCG(G; k, d)$ adopts the frugal strategy. Recall that we assume that $k = \Delta_G - s$ and $d = s + 1$, for some $s \in \{0, 1, \dots, \Delta_G - 2\}$. We begin with a result that provides a lower bound on the probability that a player, who is unhappy after a certain round, receives "enough" available colors in the next round. This will require some additional notation.

Recall that $\chi_t(v)$ denotes the color chosen by player v after round t, and that $\chi_t(\mathcal{N}(v))$ is the set of colors chosen by its neighbors after round t. For each $t \geq 1$, let $\mathcal{H}_t$ be the set of happy players after round t, and $\mathcal{U}_t = V \setminus \mathcal{H}_t$ the set of unhappy players after round t; hence, $v \in \mathcal{U}_t$ means that $\mathcal{C}_G(v; \chi_t) \geq d + 1$. Given $v \in \mathcal{U}_t$, let $\mathcal{A}_t(v) := [k] \setminus \chi_t(\mathcal{N}(v))$ be the set consisting of those colors that are not chosen by any neighbors after round t, while $\mathcal{B}_t(v) := \{\chi_t(v)\} \cup \mathcal{A}_t(v)$ be the set of available colors to v after round t; hence, under the frugal strategy, player v chooses in the next round a color uniformly at random from the set $\mathcal{B}_t(v)$. Let $p_t(v) = \frac{1}{|\mathcal{B}_t(v)|}$ be the probability with which the unhappy player v chooses a color in the next round. For $v \in \mathcal{H}_t$, set $\mathcal{B}_t(v) = \{\chi_t(v)\}$ and $p_t(v) = 1$. Similarly, given a vertex $v \in V$, let $\mathcal{H}_t(v) := \mathcal{H}_t \cap \mathcal{N}(v)$ denote the set of happy neighbors of v after round t, and let $\mathcal{F}_t(v) := \chi_t(\mathcal{H}_t(v))$ be the set of colors chosen by the happy neighbors of v after round t. Let also $\mathcal{U}_t(v) = \mathcal{N}(v) \setminus \mathcal{H}_t(v)$ denote the set consisting of the unhappy neighbors of v after round t. Observe that every color from the set $[k]\setminus\mathcal{F}_t(v)$ has a positive chance of not being chosen by the unhappy neighbors and therefore has a positive chance of belonging to the set $\mathcal{A}_{t+1}(v)$. Finally, let $f_t(v) = |\mathcal{F}_t(v)|$, and observe that, since happy players do not change their color, the sequence $\{f_t(v)\}_{t\geq 1}$ is non-decreasing. In particular, this implies that for $z \geq 1$, $|\mathcal{A}_{t+z}(v)|$ is less than or equal to $k - f_t(v)$. The next lemma establishes a lower estimate on the probability that the number of colors that are not chosen by any unhappy neighbor of player $v \in \mathcal{U}_t$ after round $t+1$ is at least $\frac{k-f_t(v)}{5^d}$.

Lemma 1. *For each $t \geq 1$ and each $v \in \mathcal{U}_t$, it holds*

$$\mathbb{P}\left(|\mathcal{A}_{t+1}(v)| \geq \frac{k - f_t(v)}{5^d}\right) \geq 1 - \frac{1-1/4^d}{1-1/5^d}. \qquad (1)$$

Proof. Let $v \in \mathcal{U}_t$ be fixed and, for simplicity in notation, let us set $f := f_t(v)$. Since v is unhappy, there are at least $d+1$ vertices $u \in \mathcal{N}(v)$ such that $\chi_t(u) = \chi_t(v)$. We now proceed with estimating $\mathbb{E}(|\mathcal{A}_{t+1}(v)|)$ from below; the proof is then completed by applying Markov's inequality. As mentioned already, any color from the set $[k]\setminus\mathcal{F}_t(v)$ has a positive chance of belonging to $\mathcal{A}_{t+1}(v)$. In particular, color $i \in [k]\setminus\mathcal{F}_t(v)$ belongs to $\mathcal{A}_{t+1}(v)$ if it is not chosen by any neighbor $u \in \mathcal{U}_t(v)$ for which $i \in \mathcal{B}_t(u)$; this happens with probability $\prod_{\{u\in\mathcal{U}_t(v): i\in\mathcal{B}_t(u)\}}(1-p_t(u))$. Therefore, denoting $E := \mathbb{E}(|\mathcal{A}_{t+1}(v)|)$, the arithmetic-geometric means inequality implies that

$$\begin{aligned} E = &\sum_{i\in[k]\setminus\mathcal{F}_t(v)} \prod_{\{u\in\mathcal{U}_t(v): i\in\mathcal{B}_t(u)\}} (1-p_t(u)) \\ \geq&(k-f)\cdot\left(\prod_{i\in[k]\setminus\mathcal{F}_t(v)} \prod_{\{u\in\mathcal{U}_t(v): i\in\mathcal{B}_t(u)\}} (1-p_t(u))\right)^{\frac{1}{k-f}} \\ \geq&(k-f)\cdot\left(\prod_{u\in\mathcal{U}_t(v)} \prod_{i\in\mathcal{B}_t(u)} (1-p_t(u))\right)^{\frac{1}{k-f}} \\ =&(k-f)\cdot\left(\prod_{u\in\mathcal{U}_t(v)} \left(1-\frac{1}{|\mathcal{B}_t(u)|}\right)^{|\mathcal{B}_t(u)|}\right)^{\frac{1}{k-f}}. \end{aligned} \qquad (2)$$

Recall that $|\mathcal{B}_t(u)| \geq 2$, for every $u \in \mathcal{U}_t(v)$. Since the sequence $\{(1-\frac{1}{m})^m\}_{m\geq 2}$ is non-decreasing, it holds $\left(1-\frac{1}{|\mathcal{B}_t(u)|}\right)^{|\mathcal{B}_t(u)|} \geq \left(1-\frac{1}{2}\right)^2 = \frac{1}{4}$. Summarizing the above, we have shown that $\mathbb{E}(|\mathcal{A}_{t+1}(v)|) \geq (k-f)\cdot\left(\frac{1}{4}\right)^{\frac{|\mathcal{U}_t(v)|}{k-f}}$.

Now, since $k = \Delta_G - d + 1$, it holds $|\mathcal{U}_t(v)| \leq \Delta_G - f = k + d - 1 - f$, and hence, we have $\frac{|\mathcal{U}_t(v)|}{k-f} \leq d$. This implies that $\left(\frac{1}{4}\right)^{\frac{|\mathcal{U}_t(v)|}{k-f}} \geq \left(\frac{1}{4}\right)^d$ and therefore, it holds $\mathbb{E}(|\mathcal{A}_{t+1}(v)|) \geq \frac{k-f}{4^d}$. To finish the proof, let $X = k - f - |\mathcal{A}_{t+1}(v)|$ and apply the previous lower bound on $\mathbb{E}(|\mathcal{A}_{t+1}(v)|)$ together with Markov's inequality to deduce $\mathbb{P}\left(|\mathcal{A}_{t+1}(v)| < \frac{k-f}{5^d}\right) = \mathbb{P}\left(X > (k-f)\cdot(1-\frac{1}{5^d})\right) < \frac{\mathbb{E}(X)}{(k-f)\cdot(1-\frac{1}{5^d})} \leq \frac{1-\frac{1}{4^d}}{1-\frac{1}{5^d}}$, as desired. □

The next lemma concerns a lower estimate of the probability that a player, who is unhappy after round t, becomes happy after two rounds.

Lemma 2. *It holds*

$$\mathbb{P}(v \in \mathcal{H}_{t+2} \,|\, v \in \mathcal{U}_t) \geq \frac{1}{2}\cdot\left(\frac{1}{4}\right)^{5^d\cdot d}\cdot\left(1-\frac{1-1/4^d}{1-1/5^d}\right). \qquad (3)$$

Proof. Let $v \in \mathcal{U}_t$. Player v will become happy after round $t+2$ if they pick a color that has been chosen by at most d neighbors. However, it could be that the set $\mathcal{B}_{t+1}(v)$ contains a *bad color*, i.e., a color that has been chosen by at least $d+1$ happy neighbors. Note that a bad color cannot make player v happy in any subsequent round. Therefore, to lower bound the probability that v is happy after round $t+2$, we may exclude bad colors from the set $\mathcal{B}_{t+1}(v)$. In the worst case, all unhappy players have a bad color in their set of available colors. Let v be an unhappy player with a bad color in their palette. The bad color is $\chi_{t+1}(v)$. Now note that, conditional on $\mathcal{A}_{t+1}(v)$ and $v \in \mathcal{U}_{t+1}$, the probability that player v is happy after round $t+2$ is greater than or equal to the probability that a fixed color from $\mathcal{B}_{t+1}(v)\setminus\{\chi_{t+1}(v)\}$ is chosen by at most d unhappy neighbors of v. Now, the probability that a fixed color $i \in \mathcal{B}_{t+1}(v)\setminus\{\chi_{t+1}(v)\}$ is chosen by at most d players $u \in \mathcal{U}_{t+1}(v)$ is greater than or equal to the probability that color i is not chosen by any player from $\mathcal{U}_{t+1}(v)$; the latter probability being equal to $\prod_{\{u\in\mathcal{U}_{t+1}(v): i\in\mathcal{B}_{t+1}(u)\}}(1-p_{t+1}(u))$. To simplify notation, let us set $\pi(u) = 1 - p_{t+1}(u)$ and $b_u = |\mathcal{B}_{t+1}(u)|$, for each player u. Further, set $S_{t+1}(v) = \mathcal{B}_{t+1}(v)\setminus\{\chi_{t+1}(v)\}$.

Then, conditional on $\mathcal{A}_{t+1}(v)$ and $v \in \mathcal{U}_{t+1}$, the probability that player v is happy after round $t+2$ is at least

$$\begin{aligned} P :=&\frac{1}{b_v}\sum_{i\in S_{t+1}(v)} \prod_{\{u\in\mathcal{U}_{t+1}(v): i\in\mathcal{B}_{t+1}(u)\}} \pi(u) \\ =&\frac{b_v-1}{b_v\cdot(b_v-1)}\sum_{i\in S_{t+1}(v)} \prod_{\{u\in\mathcal{U}_{t+1}(v): i\in\mathcal{B}_{t+1}(u)\}} \pi(u) \\ \geq&\frac{1}{2}\left(\prod_{i\in S_{t+1}(v)} \prod_{\{u\in\mathcal{U}_{t+1}(v): i\in\mathcal{B}_{t+1}(u)\}} \pi(u)\right)^{\frac{1}{b_v-1}} \\ \geq&\frac{1}{2}\left(\prod_{u\in\mathcal{U}_{t+1}(v)} \prod_{i\in\mathcal{B}_{t+1}(u)} \pi(u)\right)^{\frac{1}{b_v-1}} \\ =&\frac{1}{2}\left(\prod_{u\in\mathcal{U}_{t+1}(v)} \left(1-\frac{1}{|\mathcal{B}_{t+1}(u)|}\right)^{|\mathcal{B}_{t+1}(u)|}\right)^{\frac{1}{|\mathcal{A}_{t+1}(v)|}}, \end{aligned} \qquad (4)$$

where the first estimate follows from the arithmetic-geometric means inequality and the last equation follows from the fact that $|\mathcal{B}_{t+1}(v)| = |\mathcal{A}_{t+1}(v)| + 1$. As in the proof of Lemma 1, it holds $\left(1 - \frac{1}{|\mathcal{B}_{t+1}(u)|}\right)^{|\mathcal{B}_{t+1}(u)|} \geq \frac{1}{4}$.

Therefore, we conclude that $P \geq \frac{1}{2} \cdot \left(\frac{1}{4}\right)^{\frac{|\mathcal{U}_{t+1}(v)|}{|\mathcal{A}_{t+1}(v)|}}$. Now, observe that $|\mathcal{U}_{t+1}(v)| \leq \Delta_G - |\mathcal{H}_{t+1}(v)| = k + d - 1 - |\mathcal{H}_{t+1}(v)| \leq k + d - 1 - f_t(v)$, which implies that, conditional on the event that $|\mathcal{A}_{t+1}(v)| \geq \frac{k - f_t(v)}{5^d}$, it holds $\frac{|\mathcal{U}_{t+1}(v)|}{|\mathcal{A}_{t+1}(v)|} \leq 5^d \cdot \frac{k+d-1-f_t(v)}{k - f_t(v)} \leq 5^d \cdot d$. Hence, conditional on the event that $|\mathcal{A}_{t+1}(v)| \geq \frac{k - f_t(v)}{5^d}$, we have $P \geq \frac{1}{2} \cdot \left(\frac{1}{4}\right)^{5^d \cdot d}$, and the result follows from Lemma 1. □

We now proceed with the proof of Theorem 2.

Proof of Theorem 2. For a player $v \in V$, and a round t, let $Y_v(t)$ be random variables defined as follows:

$$Y_v(t) = \begin{cases} 1 & \text{, if } \mathcal{C}_G(v; \chi_t) \geq d + 1 \\ 0 & \text{, otherwise} \end{cases} \tag{5}$$

From Lemma 2, we know that for every player v and any round t it holds $\mathbb{P}\left(Y_{t+2}(v) = 1 \mid Y_t(v) = 1\right) \leq 1 - c_d$, where $c_d = \frac{1}{2} \cdot (\frac{1}{4})^{5^d \cdot d} \cdot (1 - \frac{1 - 1/4^d}{1 - 1/5^d})$.

Therefore, Lemma 2 implies that for any τ it holds

$$\begin{aligned} \Pi_\tau =& \mathbb{P}\left(Y_v(2\tau) = 1 \mid Y_v(0) = 1\right) \\ =& \mathbb{P}\left(\bigcap_{i=1}^{\tau} Y_v(2i) = 1 \mid Y_v(0) = 1\right) \\ =& \prod_{i=1}^{\tau} \mathbb{P}\left(Y_v(2i) = 1 \mid Y_v(2i-2) = 1\right) \\ \leq& (1 - c_d)^\tau \end{aligned} \tag{6}$$

Plugging in $\tau = \frac{1}{c_d} \cdot \log(\frac{n}{\delta})$, we deduce that for every player v, it holds

$$\mathbb{P}(Y_v(2\tau) = 1 \mid Y_v(0) = 1) \leq \frac{\delta}{n} \tag{7}$$

Now, set $Y_t = \cup_{v \in V}\{Y_v(t) = 1\}$ for an arbitrary round t. The union bound then implies that

$$\mathbb{P}(Y_\tau = 0 \mid Y_0 = 1) \geq 1 - \delta\,, \tag{8}$$

as desired. □

6. NUMERICAL EVALUATION

In this section, we evaluate the pure operation and performance of the greedy and frugal strategies in terms of the number of collisions at their convergence point and the required rounds until convergence is reached. To this end, both randomly generated graphs and graphs derived from real network topologies are analyzed in the sequel. First, different scenarios have been generated via modeling and simulation using the Erdős–Rényi random graph model $G(n, p)$ that captures the structure of any interdependent networked system in communication and computer networks. The parameter n corresponds to the number of vertices to be included in the randomly generated graph, while p is the probability with which an edge e between different vertices is created. In the following paragraphs, we consider both different numbers of vertices n and probabilities p to provide a holistic view of the algorithms' operation. The corresponding numerical results have been averaged over 100 different random graph realizations under each specific graph setting. To further provide a realistic demonstration of the operation of the proposed strategies' performance, three different real network topologies with varying numbers of nodes (vertices) and connections (edges) have also been considered, drawn from the Network Topology Zoo dataset [23]. The selected topologies span the whole range of small to large-scale networks, whose information is listed in the following paragraphs, and the respective numerical results are presented in Table 1.

First, we consider a fixed number of vertices, i.e., $n = 20$, and evaluate the concluding number of collisions and required rounds by varying the value of parameter s. Numerical results for both complete (i.e., $p = 1$) and not complete graphs with probability $p = 0.5$ are generated and averaged, and are presented in Fig. 2 and Fig. 3. The mean maximum degree of Erdős–Rényi graphs $G(20, 0.5)$ is $\Delta_G = 13$, resulting in $s \in \{0, 1, \ldots, 10\}$ and $s \in \{0, 1, \ldots, 11\}$ under the greedy and frugal strategies, respectively. Similarly, considering a mean maximum degree $\Delta_G = 19$ for the Erdős–Rényi case $G(20, 1)$, the parameter s takes values from the set $s \in \{0, 1, \ldots, 16\}$ under the greedy and $s \in \{0, 1, \ldots, 17\}$ under the frugal strategy, accordingly.

Fig. 2 depicts the number of collisions concluded in the aforementioned simulation scenarios. An increasing value of the parameter s implies a lower number of available colors at each vertex and, thus, a higher number of collisions. The important conclusion driven from Fig. 2 though is that, in this article, we have not only managed to devise a strategy that applies when the number of available colors to each player is decreased to $k = 2$, contrary to the greedy strategy proposed in [22] that is applicable when $k = 3$, but also the proposed frugal strategy performs equally or outperforms the greedy one in the majority of simulation scenarios in terms of the concluding number of collisions. The latter observation is more evident in the complete graph case (Fig. 2b) and especially for large values of the parameter s. As the value of s increases, the number of available colors $k = \Delta_G - s$ at each vertex decreases. Thus, including the color that made a

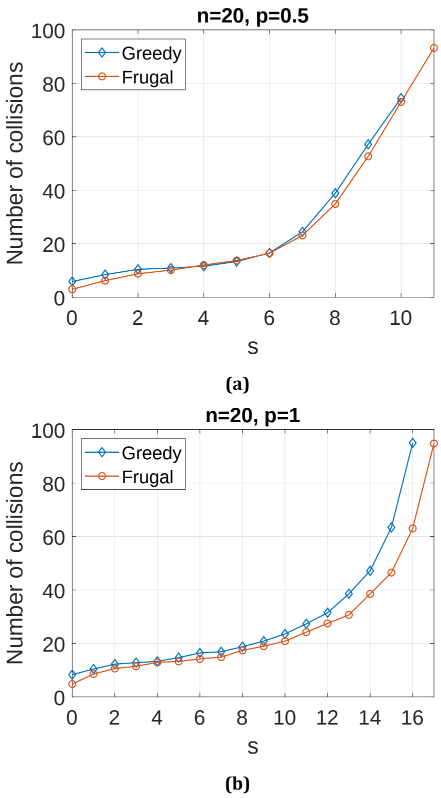

Figure 2 – Number of collisions for different values of parameter s under the randomly generated Erdős–Rényi graphs (a) $G(20, 0.5)$ and (b) $G(20, 1)$.

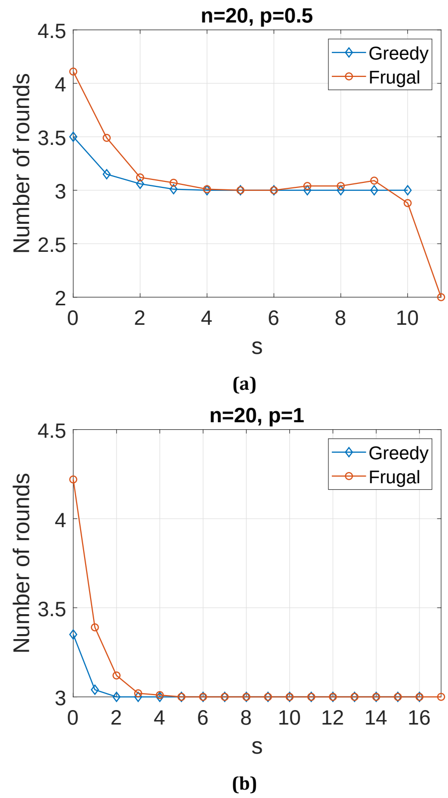

Figure 3 – Number of rounds for different values of parameter s under the randomly generated Erdős–Rényi graphs (a) $G(20, 0.5)$ and (b) $G(20, 1)$.

player unhappy at their strategy set between subsequent rounds, as suggested by the frugal strategy, is an effective method that can even reduce the overall number of collisions in the graph. For the same value of the parameter s, this simulation scenario practically validates our theoretical findings following Corollary 1 and Corollary 2 in that the collision number at each vertex under the frugal strategy is lower than that under the greedy strategy.

Continuing with Fig. 3, the mean number of rounds of the different players required under both the greedy and frugal strategies are presented for the (a) $G(20, 0.5)$ and (b) $G(20, 1)$ simulation scenarios and under varying values of s. We observe that as the value of s gets bigger, the number of rounds required for the algorithms to converge decreases owing to the vertices' threshold of acceptable collisions that becomes looser, i.e., $d = s+2$ under the greedy strategy and $d = s + 1$ under the frugal strategy. Especially for small values of s, i.e., $s < 4$, when the number of available colors is large, removing the color that made a player unhappy at the previous round from the available colors set, as suggested by the greedy strategy, allows for converging faster. For $s \geq 4$, i.e., for a highly constrained color palette, the performance of both strategies is equal, requiring on average three rounds to converge.

Subsequently, we aim to assess the performance of the greedy and frugal strategies under the randomly generated Erdős–Rényi graphs with probabilities $p = 0.5$ and

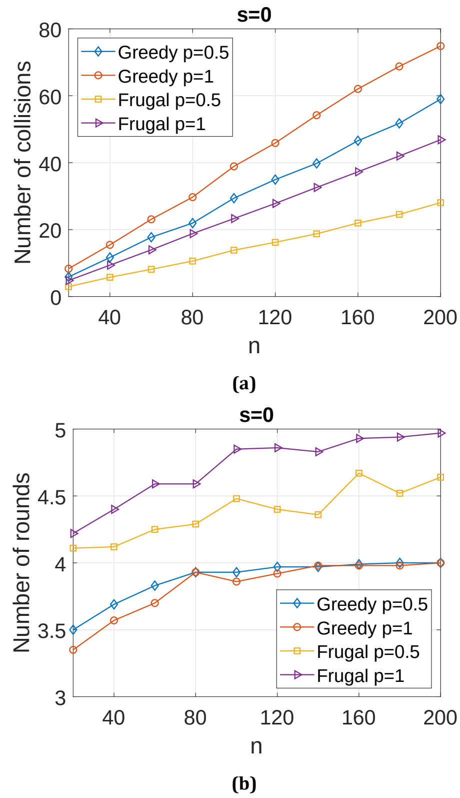

Figure 4 – Number of (a) collisions and (b) rounds for $s = 0$ and a varying number of vertices n under the randomly generated Erdős–Rényi graphs with $p = 0.5$ and $p = 1$.

$p = 1$ and varying numbers of vertices $n = [20, 200]$. For this purpose, we fix the value of the parameter s equal to $s = 0$ (see results in Fig. 4) to examine the algorithms' behavior when the number of available colors $k = \Delta_G - s$ at each vertex is the maximum possible. Fig. 4a further corroborates the fact that the frugal strategy concludes a lower number of collisions compared to the greedy one, while the gap between the two strategies becomes wider as the number of vertices in the generated graphs increases under both $p = 0.5$ and $p = 1$ cases. If we significantly increase the number of vertices equal to $n = 200$, the performance improvement of the frugal strategy is almost double, yielding half the number of collisions under the greedy strategy. Regarding the mean number of rounds of the different players required for the algorithms to converge, the greedy strategy proves to be almost one round faster in the different simulation scenarios (see Fig. 4b). As the number of vertices increases, a small increase in the required number of rounds is observed while some minor variations appear, especially under the frugal strategy, which is attributed to the randomized nature of the devised algorithms in this article. Overall, the frugal strategy introduced in this article can be even double times better in the number of collisions than our previous work, with the cost of only one extra round when the available colors at each vertex are equal to the maximum degree of the underlying graph, i.e., for $s = 0$.

Last, it is interesting to investigate the operation of the two strategies when the amount of colors available at each vertex takes the lowest possible value, which happens when $s = \Delta_G - 3$ under the greedy strategy and $s = \Delta_G - 2$ under the frugal strategy. The obtained numerical results are presented in Fig. 5 for a varying number of vertices within the range $n = [20, 200]$ and Erdős–Rényi probabilities $p = 0.5$ and $p = 1$, respectively. In this simulation scenario, the two strategies operate identically, resulting in the same number of collisions (see Fig. 5a) and the same number of rounds (see Fig. 5b) except for the Erdős–Rényi graph case with $p = 0.5$, where the players can conclude the game under lower than three rounds on average. The latter corroborates the observations driven from Fig. 2 and Fig. 3 for large values of the parameter s.

The observations derived from the preceding analysis over randomly generated graphs are further corroborated by the evaluation of three different-scale real network topologies, namely the Cernet, TW, and Colt networks [23]. These networks range from small to large scale, with 37, 71, and 146 nodes and 55, 118, and 178 connections, respectively. Additionally, the maximum degree of their underlying graphs is 12, 12, and 18, accordingly. Contrary to the already examined and randomly generated graphs, the real network topologies are characterized by highly heterogeneous and irregular underlying graphs, leading to a notable variation in the number of edges across the vertices. This means that there exist point-to-point connections between some vertices, whereas others may have much higher degrees. To assess the performance of the proposed greedy and frugal strategies under such types of underlying graphs, we have executed the corresponding algorithms and calculated the concluding number of collisions after algorithmic convergence. Given the randomized nature of the devised algorithms, the numerical results have been averaged over 100 different algorithm executions and are listed in Table 1.

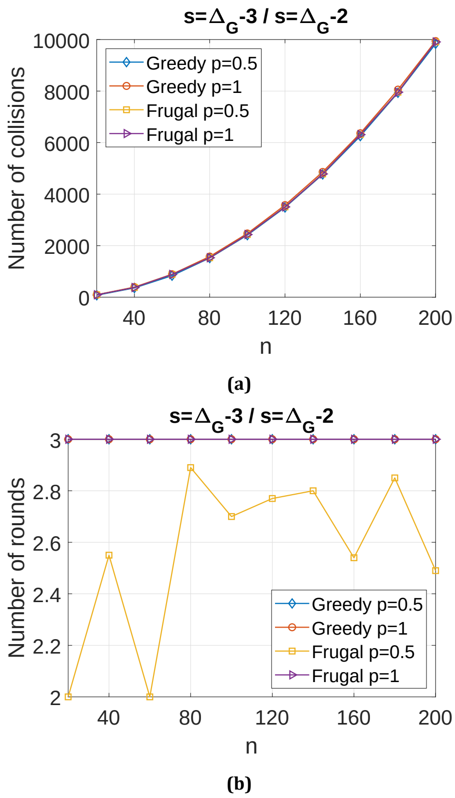

Figure 5 – Number of (a) collisions and (b) rounds for $s = \Delta_G - 3/s = \Delta_G - 2$ and a varying number of vertices n under the randomly generated Erdős–Rényi graphs with $p = 0.5$ and $p = 1$.

Specifically, Table 1 is organized into three main sections, one for each of the examined real network topologies. In each section, the concluding number of collisions by the greedy and the frugal strategies is listed along with the resulting deviation between the two strategies. The columns of Table 1 correspond to different values of the parameter s, which is $s \in \{0, \cdots, 9\}$ and $s \in \{0, \cdots, 10\}$ for the greedy and frugal strategies under the Cernet and TW networks given their maximum degree equal to 12. The respective values of s under the Colt network are $s \in \{0, \cdots, 15\}$ and $s \in \{0, \cdots, 16\}$, respectively. The circled instances within the table regard indicate cases when the frugal strategy underperforms compared to the greedy strategy, yielding more collisions than the latter. It can be observed that these cases constitute the minority, while the deviation in the number of collisions between the frugal and greedy strategies therein is small, i.e., on average

Table 1 – Number of collisions for different values of parameter s under Cernet, TW, and Colt network topologies.

	s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Cernet	Greedy	8	10	10	21	28	28	42	42	42	53	-	-	-	-	-	-	-
	Frugal	2	8	11	12	22	29	31	44	45	45	53	-	-	-	-	-	-
	Deviation	6	2	①	9	6	①	11	②	③	8	-	-	-	-	-	-	-
TW	Greedy	12	25	38	53	70	77	82	90	90	100	-	-	-	-	-	-	-
	Frugal	6	14	28	42	56	71	78	85	92	97	106	-	-	-	-	-	-
	Deviation	6	11	10	11	14	6	4	5	②	3	-	-	-	-	-	-	-
Colt	Greedy	31	63	81	94	106	112	112	121	121	132	132	132	132	132	132	132	-
	Frugal	5	33	64	82	94	107	112	113	122	122	135	135	135	136	137	140	142
	Deviation	26	30	17	12	12	5	0	8	①	10	③	③	③	④	⑤	⑧	-

one to two collisions for the Cernet and TW networks, and four collisions for the Colt network. Overall, it is evident that the frugal strategy performs remarkably better, especially as the network scale increases and for small values of the parameter s (see Colt network for $s = [0, 4]$). This is attributed to the fact that the players, i.e., network nodes, have a wider range of color options available by including the color that made them unhappy in the previous iteration. The result is fewer collisions under the frugal strategy, as explained earlier in this section.

7. CONCLUSIONS

This article provided an extended and improved version of our previous work in [22] related to the Minimum Collisions Assignment (MCA) problem in interdependent networked systems with a constrained number of resources. We considered networked systems organized in the form of a graph, where the corresponding resource allocation problem can be equivalently translated as the assignment of a finite set of colors over the vertices of the graph such that the number of collisions of vertices that are assigned the same color is minimized. Being aligned with the needs of next-generation wireless communication and computer networks, e.g., scalability, resilience, and security, we adopted game-theoretic modeling and introduced distributed, randomized algorithms that converge to a Nash equilibrium. The concluded Nash equilibrium, in turn, gave rise to a defective coloring of the underlying graph, i.e., a coloring according to which every vertex has at most a certain number of collisions. The greedy strategy, which is our initial work, is applied in cases when the color palette of each vertex contains three colors. In this article, we built upon and improved the greedy strategy to apply to cases where each vertex has only two available colors. The updated strategy was termed as frugal. The estimated number of rounds required for the frugal strategy to converge and the maximum expected number of collisions at each vertex was theoretically calculated and compared against the ones derived for the greedy in [22]. Numerical results obtained via modeling and simulation were also presented that support our theoretical analyses. Overall, the frugal strategy performed very closely to the greedy one, providing an effective method to treat resource allocation problems with a significantly constrained number of resources, i.e., colors, that could not be addressed otherwise.

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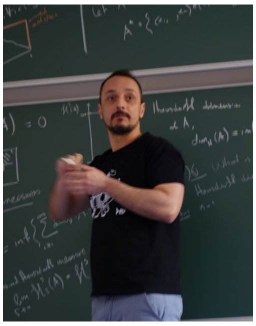

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