Full Availability Group,

Loss System

(Solutions to Exercises)

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A = traffic offered $=\lambda\cdot\tau$ erl. where λ= Calling rate = Expected no. of calls per unit of time; τ = Mean holding time expressed in the same unit of time.	TXA 1
λ = 1000 calls/hour τ = 90 sec. $A = \frac{1000\cdot 90}{3600} = \underline{\underline{25\ \text{erl.}}}$	a
λ = 1200 calls/hour τ = 2 min. $\mathbf{A} = \frac{1200\cdot 2}{60} = \underline{\underline{40\ \text{erl.}}}$	b
λ = 4 calls/sec. τ = 1.6 min. $A = 4\cdot 1.6\cdot 60 = \underline{\underline{384\ \text{erl}}}.$	c
A = 35 erl. τ = 140 sec. $\lambda = \frac{35}{140}$ calls/sec. $= \frac{35\cdot 3600}{140} = \underline{\underline{900\ \text{calls/ hour}}}$	TXA 2
A = 33 erl. λ = 1100 calls/hour $\tau = \frac{33\cdot 3600}{1100} = \underline{\underline{108\ \text{sec.}}}$	TXA 3

TXA 4 a

n = Number of devices = 10.

p = Number of busy devices.

t_p = Total time during the period with exactly p busy devices.

T = Total period. $\sum_{p=0}^{n} t_p = T$

A^1 = Traffic handled by the group during the period.

$$A^1 = \frac{1}{T} \cdot \sum_{p=0}^{n} p \cdot t_p = \sum_{p=0}^{n} p \cdot \frac{t_p}{T}$$

No. of busy devices P	Proportion of the total time with exactly P busy devices t_p/T	$p \cdot \frac{t_p}{T}$
0	—	—
1	—	—
2	—	—
3	—	—
4	—	—
5	0.10	0.50
6	0.20	1.20
7	0.25	1.75
8	0.15	1.20
9	0.20	1.80
10	0.10	1.00
Sum	$\sum \frac{t_p}{T} = 1.00$	$\sum p \cdot \frac{t_p}{T} = 7.45$

A^1 = 7.45 erl.

E = Time congestion during the period = Proportion of time, when all devices are busy

$$= \frac{t_n}{T} = \frac{t_{10}}{T} = 0.10$$

cont.

cont: TXA 4 b

Observation no.	No. of busy devices
1	8
2	8
3	**10**
4	**10**
5	9
6	7
7	7
8	6
9	5
10	5
Sum	**75**

$A^1 =$ Traffic handled $\approx \frac{75}{10} = \underline{\underline{7.5 \text{ erl.}}}$

$E =$ Time congestion $\approx \frac{2}{10} = \underline{\underline{0.20}}$

<u>Erlang Distribution</u>

TXA 5

λ = Calling rate

τ = Mean holding time

A = Traffic offered = $\lambda \cdot \tau$

E = Time congestion = Call congestion = B =

$$= E_n(A) = \frac{A^n}{n!} \Big/ \sum_{\nu=0}^{n} \frac{A^\nu}{\nu!}$$

Where n = Number of devices.
<u>n = 10</u>

Using the Erlang Table, we find:

A erl.	E_{10} (A)
1	0.0000
3	0.0008
5	0.0184
10	0.2146
15	0.4103
25	0.6224
50	0.8047
100	0.9011
200	0.9503
300	0.9668

Erlang Distribution	TXA 6

Erlang Distribution

See Example TXA 5

A = 10 erl.

Using the Erlang Table, we find:

n	E_n (10)
1	0.9091
2	0.8197
3	0.7321
5	0.5640
7	0.4090
10	0.2146
15	0.0365
20	0.0019
25	0.0000
30	0.0000

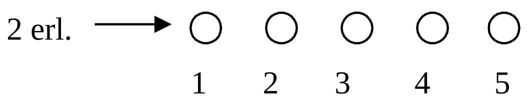
Erlang Distribution — TXA 7

E_{20} (A) = 0.005

Using the part of the Erlang Table where E is an input parameter, we find:

A = 11.092 erl.

Erlang Distribution We want to find the smallest value of n, number of devices, for which $E_n(48) \le 0.002$ Using the part of the Erlang Table where E is an input parameter, we find $E_{66}(47.51) = 0.002$ $E_{67}(48.38) = 0.002$ We conclude: n = 67 devices The other part of the Erlang Table could of course be used as well, but in that case interpolation has to be done.	TXA 8
Erlang Distribution n = Number of devices = 18 λ = Calling rate = 480 calls/hour τ = Mean holding time = 105 sec. A = Traffic offered = $\lambda \cdot \tau$ = $= \frac{480 \cdot 105}{3600} = 14$ erl. E = Time congestion B = Call congestion $E = B = E_{18}(14) = 0.0628$ (from the Erlang Table)	TXA 9 cont.

$A^1 =$ Traffic handled $= A \cdot [1 - E_n(A)] =$ $= 14 \cdot [1 - E_{18}(14)] = 14 \cdot [1 - 0.0628] = \underline{\underline{13.12 \text{ erl}}}$ a = Mean of traffic handled per device = $= \frac{A \cdot (1 - E_n(A))}{n} = \frac{A^1}{n} = \frac{13.12}{18} = \underline{\underline{0.729 \text{ erl.}}}$ Traffic rejected $= A \cdot E_n(A)$ $= 14 \cdot E_{18}(14) = 14 \cdot 0.0628 = \underline{\underline{0.88 \text{ erl.}}}$ Expected number of rejected calls per hour = $= \lambda \cdot E_n(A) = 480 \cdot E_{18}(14) = 480 \cdot 0.0628 = \underline{\underline{30.1 \text{ calls/ hour}}}$	cont: TXA 9
Erlang Distribution n = 5 devices A = 2 erl. a_ν = Traffic handled by the ν:th device (ν = 1, 2, 3, 4, 5)  Random hunting: $\alpha_\nu = \frac{A \cdot (1 - E_n(A))}{n} = \frac{2 \cdot (1 - E_5(2))}{5} =$ $= \frac{2 \cdot (1 - 0.0367)}{5} = \underline{\underline{0.385 \text{ erl.} \quad (\nu = 1,2,3,4,5)}}$	TXA 10 cont.

Sequential hunting	cont: TXA 10

Sequential hunting

$a_\nu = A \cdot [E_{\nu-1}(A) - E_\nu(A)]$

(ν = 1,2,3,4,5; E_0 (A) = 1 if A > 0)

Using the Erlang Table, we find

(E_0 (2) = 1)
E_1 (2) = 0.6667
E_2 (2) = 0.4000
E_3 (2) = 0.2105
E_4 (2) = 0.0952
E_5 (2) = 0.0367

$a_1 = 2 \cdot [E_0(2) - E_1(2)] = 2 \cdot [1 - 0.6667] = \underline{\underline{0.667 \text{ erl.}}}$

$a_2 = 2 \cdot [E_1(2) - E_2(2)] = 2 \cdot [0.6667 - 0.4000] = \underline{\underline{0.533 \text{ erl.}}}$

etc.

Result:

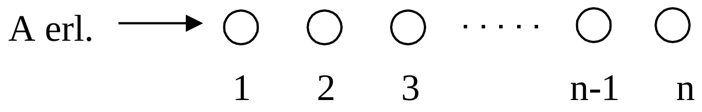
a_1 = 0.667 erl. a_2 = 0.533 erl. a_3 = 0.379 erl.
a_4 = 0.231 erl. a_5 = 0.117 erl.

Note that $\frac{1}{5} \cdot \sum_{\nu=1}^{5} a_\nu = \frac{1}{5} \cdot 1.927 = \underline{0.385 \text{ erl.}}$

We conclude that the mean value of the traffic handled per device is the same for sequential hunting and for random hunting.

Erlang Distribution

TXA 11



(sequential hunting)

a_ν = Traffic handled by the v:th device

$a_\nu = \cdot A \cdot \left[E_{\nu-1}(A) - E_\nu(A)\right]$

(ν = 1,2, ...n; $E_0(A) = 1$ if $A > 0$)

Let a = Mean of traffic handled by the different devices =

$$= \frac{1}{n} \cdot \sum_{\nu=1}^{n} a_\nu$$

$$a = \frac{1}{n} \cdot \sum_{\nu=1}^{n} A \cdot \left[E_{\nu-1}(A) - E_\nu(A)\right] =$$

$$= \frac{1}{n} \cdot A \cdot \left[1 + \sum_{\nu=1}^{n-1} E_\nu(A) - \sum_{\nu=1}^{n-1} E_\nu(A) - E_n(A)\right] =$$

$$= \frac{A \cdot \left[1 - E_n(A)\right]}{n} =$$

= Traffic handled per device in a group with random hunting, n devices and traffic offered = A erl.

Erlang Distribution

TXA 12

E_{10} (A) = 0.005 gives A = 3.961 erl.
E_{100} (A) = 0.005 gives A = 80.91 erl.

(according to the Erlang Table)

	n = 10	n = 100
A	3.961 erl.	80.91 erl.
E_n (A)	0.005	0.005
$A_1 = 1.1 \cdot A$	4.36 erl.	89.00 erl.
E_n (A_1)	0.009	0.024
$A_2 = 1.2 \cdot A$	4.75 erl.	97.09 erl
E_n (A_2)	0.014	0.059

If “a” denotes traffic handled per device, we find:

	n = 10	n = 100
A	a = 0.394	a = 0.805
$A_1 = 1.1 \cdot A$	a = 0.432	a = 0.869
$A_2 = 1.2 \cdot A$	a = 0.469	a = 0.914

Denote the length of the interval by T. The distribution function F(x) is by definition: TXA 13

$$F(x) = P\{T \leq x\}$$

Calculate the complementary distribution function:

$$1 - F(x) = P\{T > x\}$$

$0 < j < n$: a

The interval may end by a call or a termination. Then T > x if

1) No call in (0, x) and
2) No termination in (0, x).

The probability of 1) is $e^{-\lambda \cdot x}$

The probability of 2) is $e^{-j \cdot \frac{x}{\tau}}$

Consequently,

$$1 - F(x) = e^{-\lambda \cdot x} \cdot e^{-j \cdot \frac{x}{\tau}} = e^{-\frac{A+j}{\tau} \cdot x}$$

$$F(x) = 1 - e^{-\frac{A+j}{\tau} \cdot x}$$

i.e. an exponential distribution with the parameter $\frac{A+j}{\tau}$

The mean of an exponentially distributed random variable is equal to the inverse of the parameter i.e. b

$$\frac{\tau}{A+j}$$

cont.

cont: TXA 13 c

The state ends by a transition to (j+1) if the first event is a call.

The conditional probability that the first event is a call occuring in (x, x+dx) is

$$P\{\text{the first event is a call}/\text{a call occurs in } (x, x+dx)\} =$$
$$P\{\text{no termination in } (0,x),\ \text{no call in } (0,x),\ \text{one call in } (x, x+dx)\} =$$
$$e^{-j\cdot\frac{x}{\tau}} \cdot e^{-\lambda\cdot x} \cdot \lambda \cdot dx$$

Taking away the condition, by the theorem of total probaility, the required probability p is

$$p = \int_0^\infty e^{-j\cdot\frac{x}{\tau}} \cdot e^{-\lambda\cdot x} \cdot \lambda \cdot dx = \frac{\lambda}{\frac{A+j}{\tau}} = \underline{\underline{\frac{A}{A+j}}}$$

The probability that the state ends by a transition to (j-1) is
1-p i.e.

$$1 - \frac{A}{A+j} = \underline{\underline{\frac{j}{A+j}}}$$

$\underline{\underline{j = 0}}$:

$1 - F(x) = P\{\text{no call in } (0,x)\} = e^{-\lambda\cdot x}$ a

$F(x) = \underline{\underline{1 - e^{-\lambda\cdot x}}}$

The mean value is the inverse of the parameter = b

$$\underline{\underline{\frac{1}{\lambda}}}$$

cont.

$P\{state(j=0) \rightarrow state(j=1)\} = \underline{\underline{1}}$ $P\{state(j=0) \rightarrow state(j=-1)\} = \underline{\underline{0}}$ $\underline{\underline{j=n:}}$ $1-F(x) = P\{\text{no termination in } (0,x)\} = e^{-n \cdot \frac{x}{\tau}}$ $F(x) = \underline{\underline{1-e^{-\frac{n \cdot x}{\tau}}}}$ Mean value = $\underline{\underline{\frac{\tau}{n}}}$ $P\{state(j=n) \rightarrow state(j=n+1)\} = \underline{\underline{0}}$ $P\{state(j=n) \rightarrow state(j=n-1)\} = \underline{\underline{1}}$	cont: TXA 13 c a b c
Denote the call length by X. $\underline{\text{Exactly 2 pulses if } 3 \le x < 6:}$ $P\{3 \le x < 6\} = F(6) - F(3) =$ $= \left(1-e^{-\frac{6}{3}}\right) - \left(1-e^{-\frac{3}{3}}\right) =$ $= e^{-1} - e^{-2} = 0.3679 - 0.1353 = \underline{\underline{0.2326}}$ $\underline{< 2 \text{ pulses if } x < 3:}$ $P\{x < 3\} = 1 - e^{-\frac{3}{3}} = 1 - e^{-1} = \underline{\underline{0.6321}}$ $\underline{> 2 \text{ pulses if } x \ge 6:}$ $P\{x \ge 6\} = e^{-\frac{6}{3}} = e^{-2} = \underline{\underline{0.1353}}$	TXA 14 a b c cont.

cont: TXA 14 d

<u>≥ 2 pulses if x ≥ 3:</u>

$$P\{x \geq 3\} = e^{-\frac{3}{3}} = e^{-1} = \underline{\underline{0.3679}}$$

e

<u>≤ 2 pulses if x < 6:</u>

$$P\{x < 6\} = 1 - e^{-\frac{6}{3}} = 1 - e^{-2} = \underline{\underline{0.8647}}$$

TXA 15

Denote the number of pulses by K.

K	Probability
1	$1 - e^{-\frac{m}{\tau}}$
2	$e^{-\frac{m}{\tau}} - e^{-\frac{2m}{\tau}}$
3	$e^{-\frac{2m}{\tau}} - e^{-\frac{3m}{\tau}}$
⋮	⋮
j	$e^{-\frac{(j-1)\cdot m}{\tau}} - e^{-\frac{j\cdot m}{\tau}}$
⋮	⋮

$$E\{K\} = \sum_{j=1}^{\infty} j \cdot \left(e^{-\frac{(j-1)\cdot m}{\tau}} - e^{-\frac{j\cdot m}{\tau}} \right) =$$

$$= \left(1 - e^{-\frac{m}{\tau}}\right) + 2\cdot\left(e^{-\frac{m}{\tau}} - e^{-\frac{2m}{\tau}}\right) + 3\cdot\left(e^{-\frac{2m}{\tau}} - e^{-\frac{3m}{\tau}}\right) + \ldots$$

$$= 1 \underbrace{- e^{-\frac{m}{\tau}} + 2\cdot e^{-\frac{m}{\tau}}} \underbrace{- 2\cdot e^{-\frac{2m}{\tau}} + 3\cdot e^{-\frac{2m}{\tau}}} \underbrace{- 3\cdot e^{-\frac{3m}{\tau}} + \ldots}$$

$$= 1 + e^{-\frac{m}{\tau}} + e^{-\frac{2m}{\tau}} + e^{-\frac{3m}{\tau}} + \ldots$$

= a geometric series with quotient $e^{-\frac{m}{\tau}}$,

so the sum is $\dfrac{1}{1 - e^{-\frac{m}{\tau}}} = \dfrac{e^{m/\tau}}{e^{m/\tau} - 1}$

Denote length of conversation by x. $P\{x > 6\,\text{min.}\} = P\{x > 6 \ \underline{\text{and}}\ A \to B\} +$ $+ P\{x > 6 \ \underline{\text{and}}\ B \to A\} =$ $= P_{A\to B} \cdot P\{x > 6 / A \to B\} + P_{B\to A} \cdot P\{x > 6 / B \to A\} =$ $= 0.55 \cdot e^{-\frac{6}{4}} + 0.45 \cdot e^{-\frac{6}{3}} = 0.55 \cdot e^{-1.5} + 0.45 \cdot e^{-2} =$ $= 0.55 \cdot 0.2231 + 0.45 \cdot 0.1353 = \underline{\underline{0.184}}$	TXA 16
Take one of the conversations. The probability that it is still in progress after 1 min. is $e^{-\frac{1}{3}}$ The probability that it has ended before 1 min. is $1 - e^{-\frac{1}{3}}$ The five conversations are independent of each other. So the problem is analog to the situation when we make 5 independent trials in which the probability of success is constant. In this example success = conversation in progress, so by the binomial distribution the required probability is $\binom{5}{2} \cdot \left(e^{-\frac{1}{3}}\right)^2 \cdot \left(1 - e^{-\frac{1}{3}}\right)^{5-2} =$ $= 10 \cdot 0.7165^2 \cdot 0.2835^3 = \underline{\underline{0.117}}$	TXA 17

$a_9 = \underline{\underline{A\left(E_8(A) - E_9(A)\right)}}$	TXA 18 a
$P_9 = \frac{A^9}{9!} \Big/ \sum_{j=0}^{10} \frac{A^j}{j!} = \underline{\underline{\frac{10}{A} \cdot E_{10}(A)}}$	b
$\underline{\underline{E_9(A)}}$	c
$P\{9 \text{ first trunks occupied}\} =$ $= P\{9 \text{ first occupied } \underline{\text{and}} \text{ 10th free}\} +$ $+ P\{9 \text{ first occupied } \underline{\text{and}} \text{ 10th occupied}\}$ Left side $= E_9(A)$. Last term on RHS $= P\{\text{all 10 occupied}\} = E_{10}(A)$. Consequently: $E_9(A) = P(9 \text{ first occ. and 10th free}) + E_{10}(A)$ and $P\{9 \text{ first occ. and 10th free}\} = E_9(A) - E_{10}(A)$	d

As the number of devices is the same as the number of sources there is no interference between the calls from different sources. The offered traffic will be carried. So the problem is analogous to a series of 6 experiments where the probability of success in each experiment is equal to the probability of finding a source being occupied = 0.5 (as each source offeres 0.5 erl.) $\underline{\underline{\text{Traffic offered}}} = 6 \cdot 0.5 = \underline{\underline{3 \text{ erl.}}}$ $\underline{\underline{\text{Time congestion}}} = P\{\text{all 6 trunks are occupied}\} =$ $= P\{\text{all 6 sources are occupied}\} = 0.5^6 = \underline{\underline{0.016}}$ $\underline{\underline{\text{Call congestion} = 0}}$ as no calls can arrive when all trunks are occupied.	TXA 19
N = No. of sources. n=No. of devices. N >> n ⇒Erlang Distribution: A = 3 erl. $\underline{\underline{\text{Time congestion} = \text{Call congestion} =}}$ $= E_n(A) = E_6(3) = \underline{\underline{0.052}}$	TXA 20
(see table below)	TXA 21

No. of Sources N	No. of devices n	Distribution	Traffic offered per source erl.	Traffic offered erl.	Call Conges -tion B	Time Conges-tion E
6	6	BERNOULLI	0.5	3	0	0.016
∞	6	ERLANG	0	3	0.052	0.052

TXA 22

a

Possible states

Device no.	0 busy devices	1 busy device	2 busy devices	3 busy devices	4 busy devices
1	○	● ○ ○ ○	● ● ● ○ ○ ○	● ● ● ○	●
2	○	○ ● ○ ○	● ○ ○ ● ● ○	● ● ○ ●	●
3	○	○ ○ ● ○	○ ● ○ ● ○ ●	● ○ ● ●	●
4	○	○ ○ ○ ●	○ ○ ● ○ ● ●	○ ● ● ●	●

b

No. of states with exactly 0 busy devices = 1
-“- -“- -“- -“- -“- 1 -“- -“- = 4
-“- -“- -“- -“- -“- 2 -“- -“- = 6
-“- -“- -“- -“- -“- 3 -“- -“- = 4
-“- -“- -“- -“- -“- 4 -“- -“- = 1

c

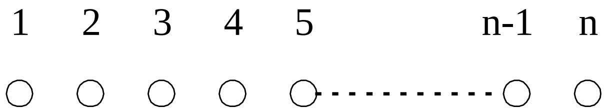
Total no. of different states =

= 1 + 4 + 6 + 4 + 1 = 16

d

No. of states, where devices no. 1 and no. 3 are busy = 4

TXA 23

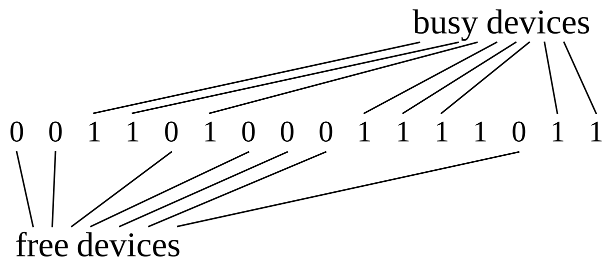


The different states are (0), (1), (2) ..., (n), where (p) denotes p busy devices (p = 0,1,...n). a

The number of different states are thus $= n + 1$.

We may consider a sequence of n symbols, each of which is either 0 or 1, where 0 denotes a free device and 1 a busy device. b

Example:



We have 2 possibilities for each device, 0 or 1.

Therefore there are $\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n\ \text{factors}} = 2^n$ different sequences

and so 2^n different states.

The number of different states with exactly p devices engaged is the same as the number of combinations of **p** out of **n** elements. c

Thus, the number of different states with exactly p devices engaged is

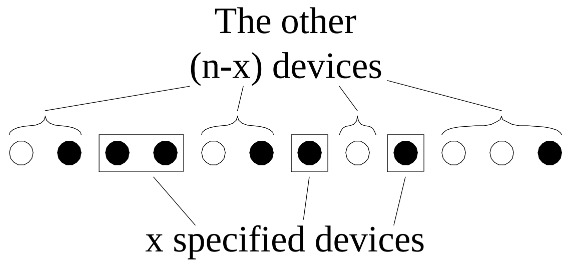
$$\binom{n}{p} \qquad (p = 0,\ 1,\ \ldots,\ n)$$

Note:

$$\sum_{p=0}^{n} \binom{n}{p} = (1+1)^n = 2^n \text{(compare with b)}$$

cont.

cont: TXA 23 d



In all p devices engaged, where $p \geq x$. Among the (n-x) devices, there are thus (p-x) devices engaged. Analogously to c) we can then state:

The number of different states with x specified devices engaged and, simultaneously, exactly p devices engaged is

$$\binom{n-x}{p-x}$$

$$(p = x, x+1, \ldots, n; \quad x = 0, 1, \ldots, n)$$

e

$$[p] = P\{\text{exactly any p devices being engaged}\}.$$

$$(p = 0, 1, \ldots, n).$$

A specified state with exactly p devices engaged has the probability

$$\frac{[p]}{\binom{n}{p}}$$

if we have random hunting, as there are in all $\binom{n}{p}$ different states with exactly p devices engaged according to c) and as all those different states will have one and the same probability.

cont.

In order to find	cont: TXA 23 e

In order to find

$$h(x,p) = P\{x \text{ specified devices engaged, and, simultaneously, exactly } p \text{ devices engaged}\}$$

we shall thus add a certain number of equal probabilities,

namely $\dfrac{[p]}{\binom{n}{p}}$; that number is, according to d), $\binom{n-x}{p-x}$.

Therefore,

$$h(x,p) = \binom{n-x}{p-x} \cdot \frac{[p]}{\binom{n}{p}} \qquad (p = x, x+1, \ldots, n)$$

Now, we have that

$$H(x) = P\{x \text{ specified devices engaged}\}$$

must be equal to

$$H(x) = \sum_{p=x}^{n} h(x,p)$$

so the result is:

$$H(x) = \sum_{p=x}^{n} \binom{n-x}{p-x} \cdot \frac{[p]}{\binom{n}{p}}$$

cont: TXA 23 e

The probability that 3 specified devices are engaged = the probability that 3 specified sources are engaged = The probability of success in 3 specified trials out of 6 independent experiments where the probability of success = 0.5 in each trial, i.e. $$\underline{\underline{0.5^3}} = 0.125$$	TXA 24
Erlang Distribution: $$H(x) = \frac{E_n(A)}{E_{n-x}(A)}$$ n = 6 ; A = 3 erl; x = 3 $$H(3) = \frac{E_6(3)}{E_3(3)} = \frac{0.052157}{0.346154} = \underline{\underline{0.151}}$$	TXA 25