RECOMMENDATION ITU-R SM.1235-0*

PERFORMANCE FUNCTIONS FOR DIGITAL MODULATION SYSTEMS IN AN INTERFERENCE ENVIRONMENT

(Questions ITU-R 44/1 and ITU-R 45/1)

(1997)

Scope

This Recommendation serves as a basis for the performance estimation functions of various digital modulation systems, receiving interference from one emitter.

Keywords

Digital modulation, undesired signal, channel interface, signal-to-noise ratio

The ITU Radiocommunication Assembly,

considering

a) that the value of the performance function at the receiver input for various combinations of modulation types of interference and desired signals essentially can define spectrum utilization efficiency;

b) that performance functions depend on the criteria for estimation of the signal reception quality and the modulation types of the interference and desired signals;

c) that performance functions can be defined either experimentally, graphically or calculated by means of formulae,

recommends

1 that for the performance estimation of various digital modulation systems, receiving interference from one emitter, calculated graphs presented in Annex 1 should be used;

2 that for the performance estimation of multiple digital phase shift keying (MPSK) systems, receiving interference from one or more emitters, calculated graphs or the analytical method presented in Annex 2 should be used.

ANNEX 1

Performance function for various digital modulation systems with only one interfering system

1 Digital receiver model

A simplified model of a communications receiver is shown in Fig. 1. The input to the channel interface is the superposition of the desired and undesired signals appearing at the receiver antenna output. The channel interface is composed of a number of circuit elements and is characterized by a receiver selectivity and by desired and undesired signal characteristics. Several Reports provide means to determine the nature of desired and undesired signals at the input to the demodulator given channel interface characteristics. The most important channel interface characteristics to consider are the bandwidth relationship between the undesired signal and the channel interface, off-tuning between the receiver and the undesired signal, and non-linear effects.

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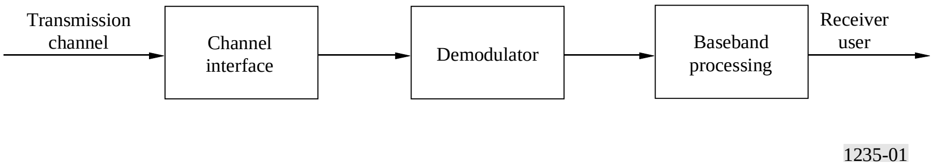
* Radiocommunication Study Group 1 made editorial amendments to this Recommendation in the year 2019 in accordance with Resolution ITU-R 1.

FIGURE 1

General communication system receiver model



Undesired signals are characterized as follows:

– *Undistorted*:

The ideal waveform transmitted by the interfering transmitter. The signal may be specified in the frequency domain in terms of power spectral density.

– *Noiselike*:

The signal varies in amplitude according to a normal (Gaussian) distribution. The signal may have a flat spectrum and is referred to as additive white Gaussian noise (AWGN).

– *Continuous wave (CW)*:

A constant frequency sinusoid whose phase with respect to the receiver is assumed to be a uniformly distributed random variable.

– *Impulsive*:

A sequence of periodic or randomly spaced pulses, each of which is of short duration compared to the time between pulses.

Undesired signals may be either continuous or intermittent. An intermittent undesired signal may be defined as a signal whose statistics such as amplitude distribution function, mean and variance are time-varying when observed at a victim receiver. Interference due to a co-located frequency hopper is an example of an intermittent undesired signal in the sense that the victim receiver will typically exhibit time-varying performance degradation. The recommended analysis procedure for the case of intermittent undesired signals involves partitioning the observation interval into contiguous time segments, or epochs, during each of which the undesired signal statistics are (approximately) constant. A separate degradation analysis is performed for each epoch, and the results are time-averaged. It is important that the time-averaging not be performed on signals until they have been demodulated.

For electromagnetic compatibility (EMC) analyses using the performance curves in this Recommendation, the undesired signal at the receiver input can usually be assumed to be either undistorted (i.e., the output of an interfering transmitter with known waveform characteristics) or noiselike. The channel interface characteristics are then used to determine the undesired signal at the demodulator input. The curves show the demodulator output bit error rate as a function of the ratio of the desired symbol energy-to-noise power spectral density (E/N_0) or the ratio of the desired symbol energy-to-interference energy (E/I_e) at the demodulator input. The noise is assumed to be Gaussian, and the interference is assumed to be continuous-wave. The analyst must determine whether the undesired signal at the input to the demodulator more closely resembles noise or CW interference or some combination of the two. This determination may include a prediction regarding the nature of the interfering signal spectrum at the demodulator input based on the passband of the channel interface and the interfering signal RF characteristics.

The remainder of the Recommendation addresses the individual sections of the receiver model shown in Fig. 1 following the channel interface. The output of any particular section may be found by concatenating the effects of that section and any preceding sections.

2 Performance of digital demodulators

Typical M-ary digital demodulator performance is given in terms of P_s versus E/N_0 and E/I. These terms are defined as follows:

M: number of possible distinct symbols. For binary signalling, $M = 2$

P_s: symbol-error probability. The bit error probability P_b, which is also often used, cannot exceed P_s. When $M = 2$, $P_b = P_s$

E/N_0: ratio of average signal energy (J)-to-noise power spectral density (W/Hz) as specified at the demodulator input (dB)

E/I_e: ratio of average signal energy per symbol (or per bit)-to-interference energy per symbol (or per bit), as specified at the demodulator input (dB).

Power ratios, in particular the signal-to-noise ratio (S/N), may be used instead of energy ratios by noting that:

$$E/N_0 = (S/N)\,(B\,T) \tag{1}$$

where:

- B: receiver bandwidth (Hz)
- T: symbol duration (s)
- S/N: measured at the demodulator input.

Table 1 summarizes the types of modulation presented, which curve to use for each, and the undesired signal (CW or noise) for which probabilities of bit or symbol errors may be obtained. The curves identified in Table 1 are provided in Figs. 2 through 24. These curves relate receiver performance in terms of symbol or bit error rate in the presence of noise and/or interference. The noise is assumed to be Gaussian, and the interference is assumed to be CW. The curves have been developed assuming optimum receiver design, i.e., bandwidths associated with the demodulator were designed for the associated system bit durations and data rates.

TABLE 1

Summary of digital modulation types considered

Modulation type	Plot of	Interfering signal [1]	Figure number
CPSK, M-ary	P_s versus E_b/N_0	N	2
CPSK, M = 2	P_s versus E/N_0, E/I_e	I_e	3
CPSK, M = 4	P_s versus E/N_0, E/I_e	I_e	4
CPSK, M = 8	P_s versus E/N_0, E/I_e	I_e	5
CPSK, M = 16	P_s versus E/N_0, E/I_e	I_e	6
QPSK	P_s versus E/N_0, E/I_e	I_e	4
O-QPSK (offset-QPSK)	P_s versus E/N_0, E/I_e	I_e	4
DPSK, M = 2	P_s versus E/N_0, E/I_e	N, I_e	7
DPSK, M = 4	P_s versus E/N_0, E/I_e	N, I_e	8
DPSK, M = 8, 16	P_s versus E/N_0, E/I_e	I_e	9
CFSK, M-ary	P_b versus E_b/N_0	N	10
CFSK, M = 2	P_s versus E/N_0, E/I_e	I_e	11
MSK	P_s versus E/N_0, E/I_e	I_e	4
NCFSK, M-ary	P_s versus E_b/N_0	N	12
NCFSK, M = 2	P_s versus E/N_0, E/I_e (interference in one channel)	I_e	13
NCFSK, M = 2	P_s versus E/N_0, E/I_e (equal interfering tones in each channel)	I_e	14
NCFSK, M = 4	P_s versus E/N_0, E/I_e	I_e	15
NCFSK, M = 8	P_s versus E/N_0, E/I_e	I_e	16
CASK, M-ary, bipolar	P_s versus E_b/N_0	N	17
CASK, M = 16, bipolar	P_s versus E/N_0, E/I_e	I_e	18
CASK, M-ary, unipolar	P_s versus E_b/N_0	N	19
CASK, M = 2, unipolar (also called OOK or on off keying)	P_s versus E_b/N_0, E/I_e	I_e	20
NCASK, M-ary	P_s versus E_b/N_0	N	21
QAM, M-ary (quadrature amplitude modulation)	P_s versus E_b/N_0	N	22
QAM, M = 4	P_s versus E/N_0, E/I_e	I_e	4
QAM, M = 16	P_s versus E/N_0, E/I_e	I_e	23
QAM, M = 64	P_s versus E/N_0, E/I_e	I_e	24

[1] N: AWGN; I_e: CW interference.

When using the curves, be alert to the parameters used. Bit energy-to-noise ratio (E_b/N_0) rather than symbol energy-to-noise (E/N_0) was used in most of the figures comparing M-ary schemes for different values of M in order to simplify the graphs. The value E represents the average symbol energy. The relationship between symbol energy and the equivalent bit energy is:

$$E = E_b \log_2 M \tag{2}$$

Some of the figures presented have the probability of a bit error, P_b, rather than the probability of a symbol error, P_s. The relationship between the two, for orthogonal signalling (coherent frequency shift keying (CFSK), non-coherent frequency shift keying (NCFSK), minimum shift keying (MSK)), is:

$$P_b = \frac{2^{k-1}}{2^k - 1} P_s \tag{3}$$

where $k = \log_2 M$ (number is equivalent bits).

For M-ary coherent phase shift keying (CPSK) and differential phase shift keying (DPSK) (Gray encoding is assumed) and for coherent amplitude shift keying (CASK) and non-coherent amplitude shift keying (NCASK) the relationship is:

$$P_b = P_s / k \tag{4}$$

FIGURE 2

P_s versus E_b/N_0 for M-ary CPSK

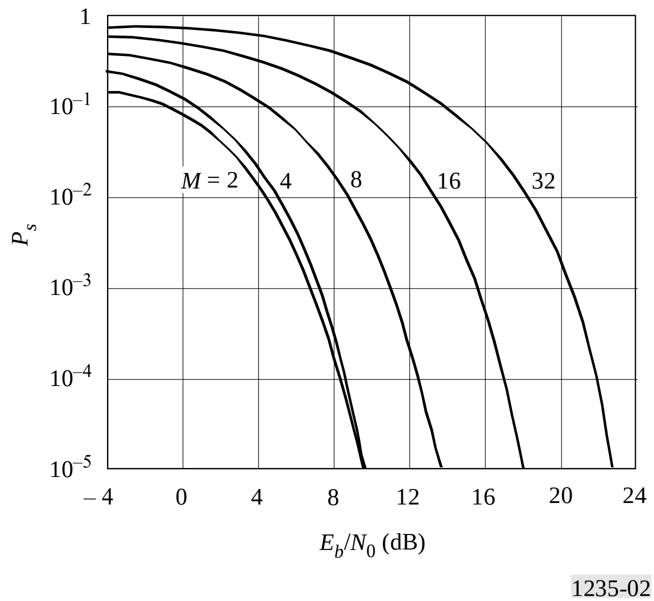


FIGURE 3

P_s versus E/N_0 and E/I_e for binary CPSK

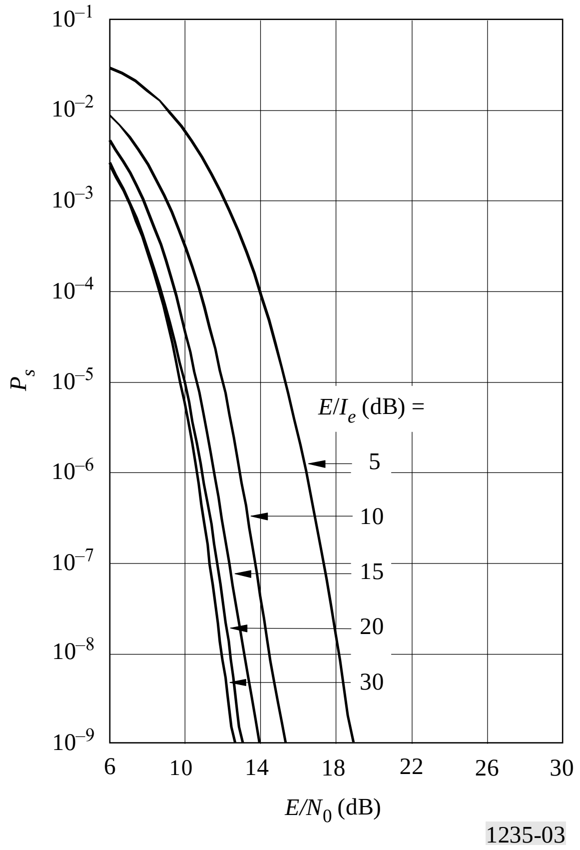


FIGURE 4

P_s versus E/N_0 and E/I_e for 4-ary CPSK, QPSK, O-QPSK, MSK and 4-ary QAM

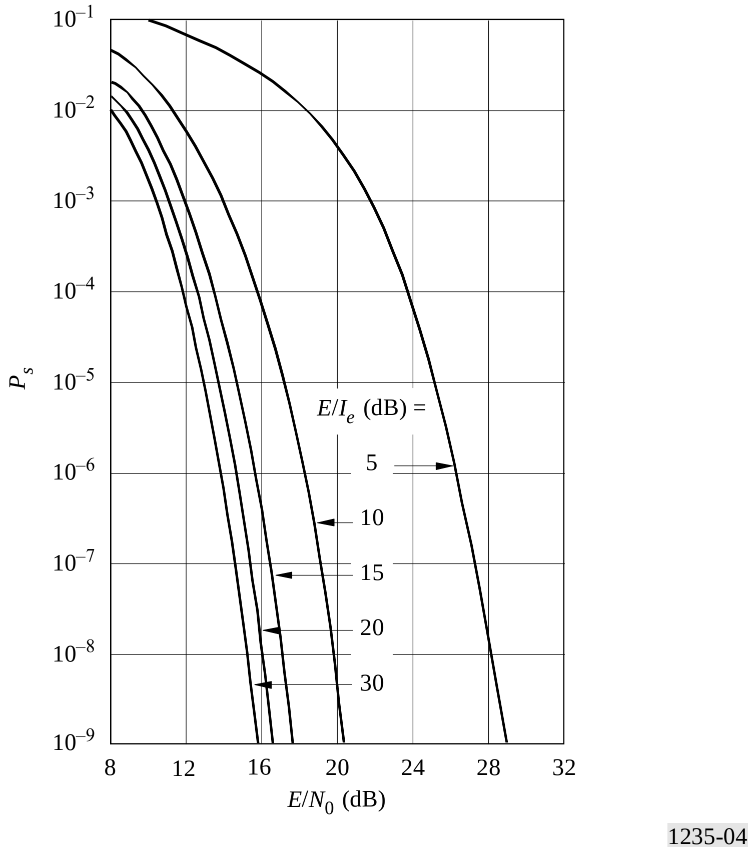


FIGURE 5

P_s versus E/N_0 and E/I_e for 8-ary CPSK

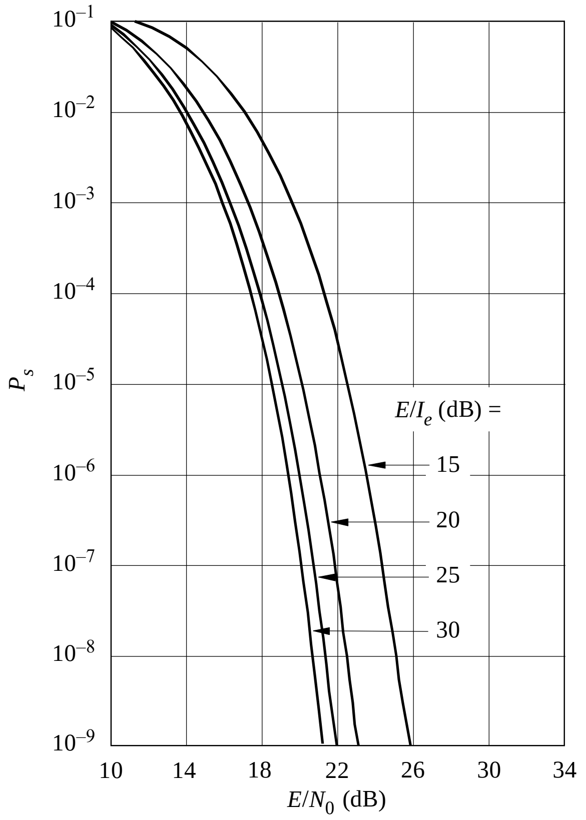


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FIGURE 6

P_s versus E/N_0 and E/I_e for 16-ary CPSK

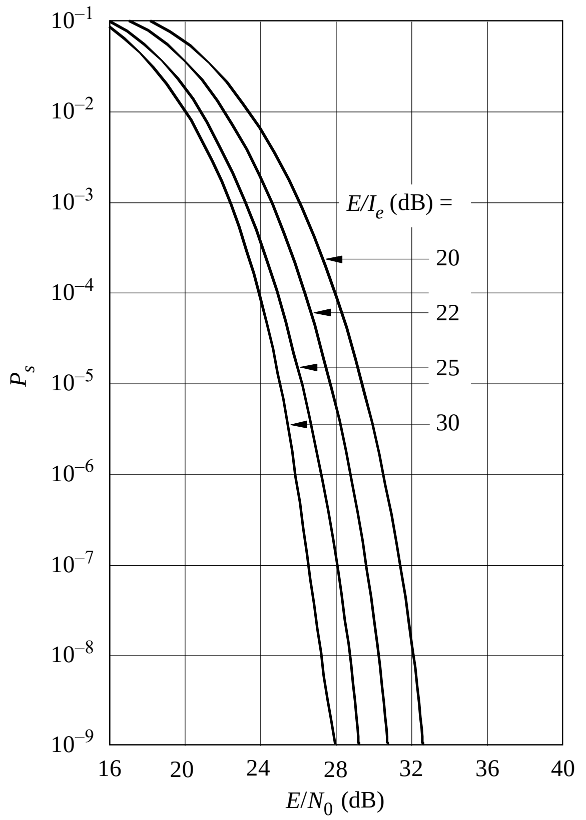

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FIGURE 7

P_s versus E/N_0 and E/I_e for binary DPSK

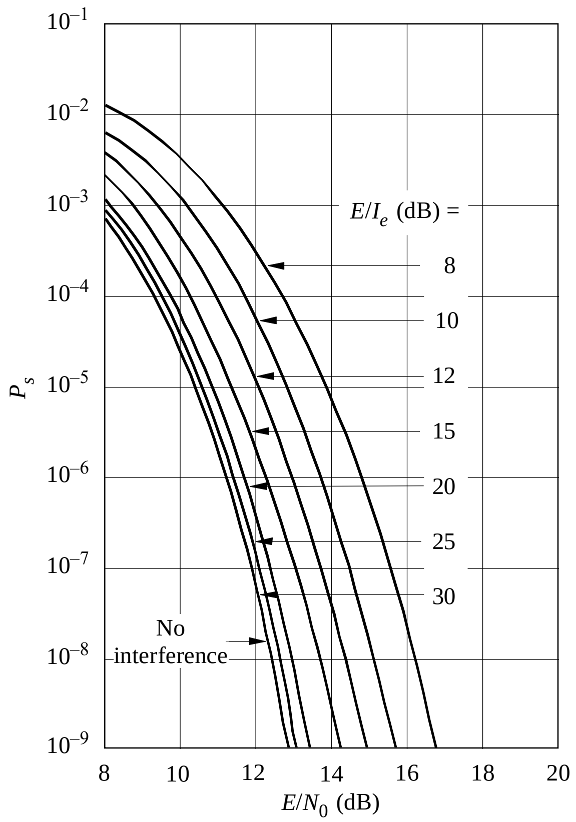

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FIGURE 8

P_s versus E/N_0 and E/I_e for 4-ary DPSK

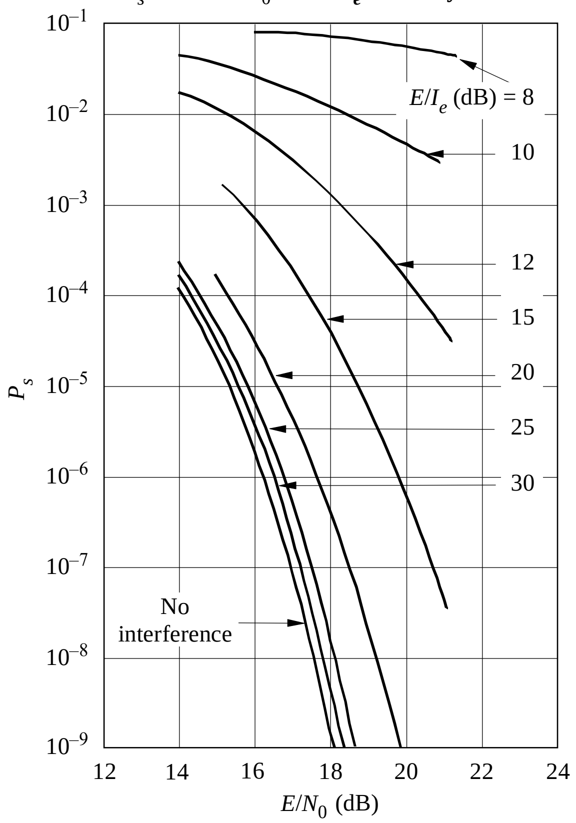

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FIGURE 9

P_s versus E/N_0 and E/I_e for 8-ary and 16-ary DPSK

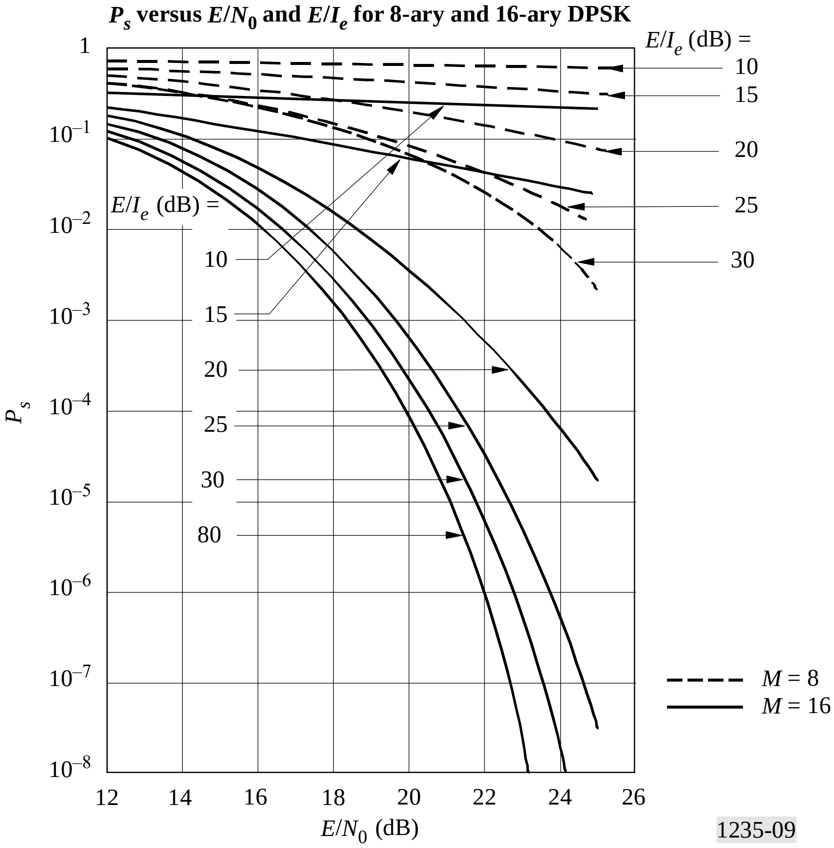

FIGURE 10

P_b versus E_b/N_0 for *M*-ary CFSK

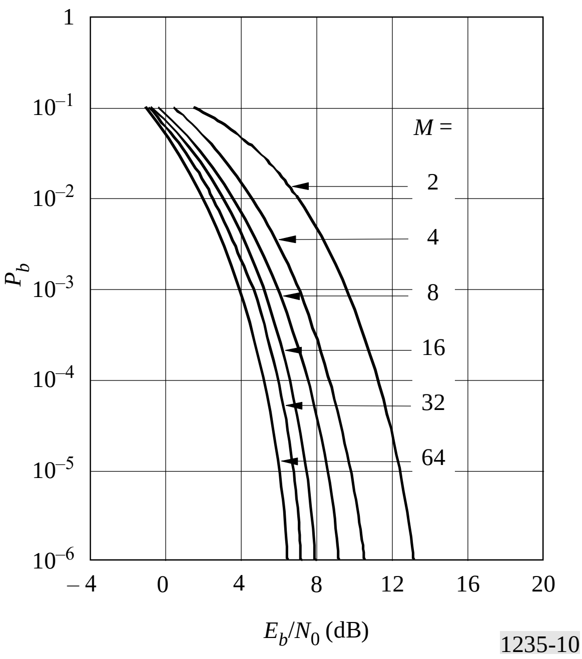

FIGURE 11

P_s versus E/N_0 and E/I_e for binary CFSK

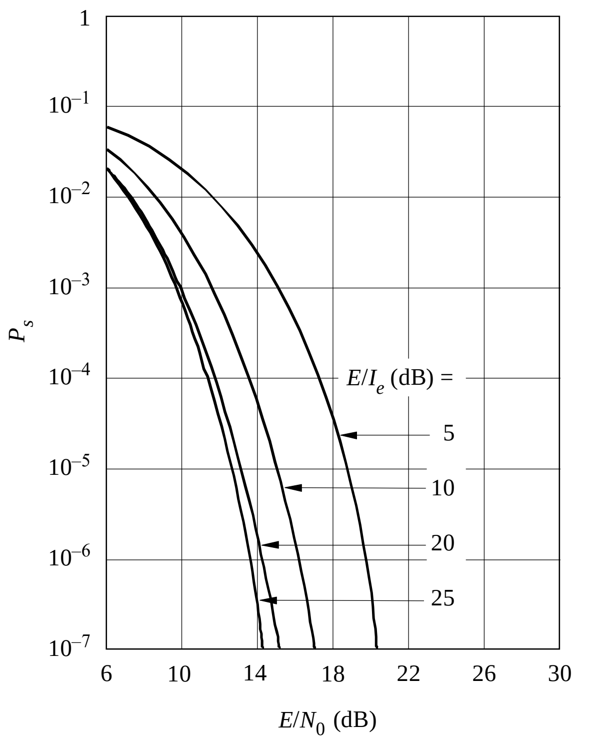

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FIGURE 12

P_s versus E_b/N_0 for M-ary NCFSK

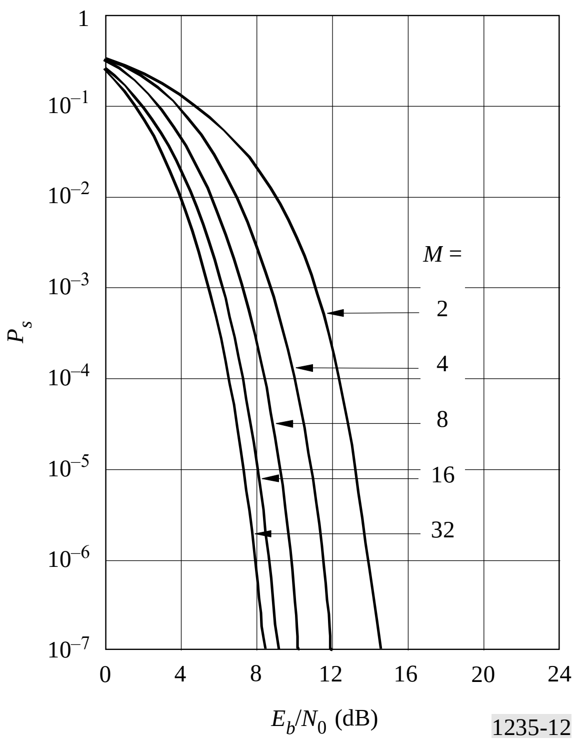

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FIGURE 13

P_s versus E/N_0 and E/I_e for binary NCFSK with interfering in one channel

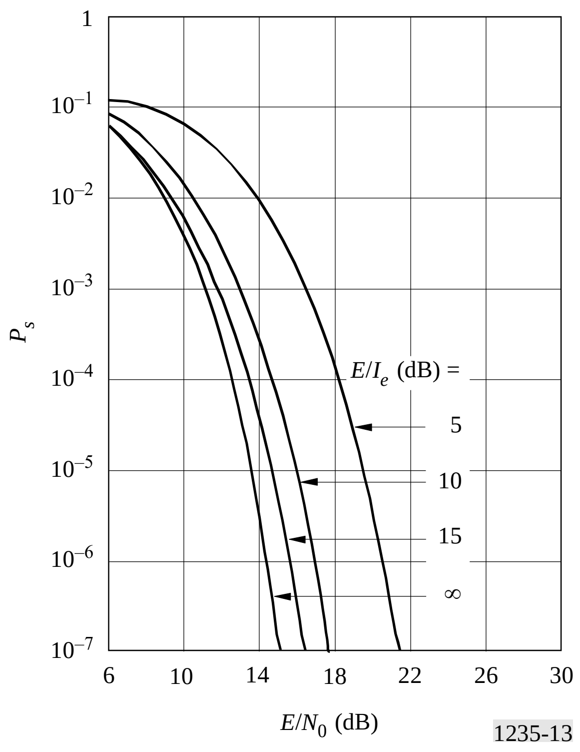

FIGURE 14

P_s versus E/N_0 and E/I_e for binary NCFSK with equal interfering tones in each channel

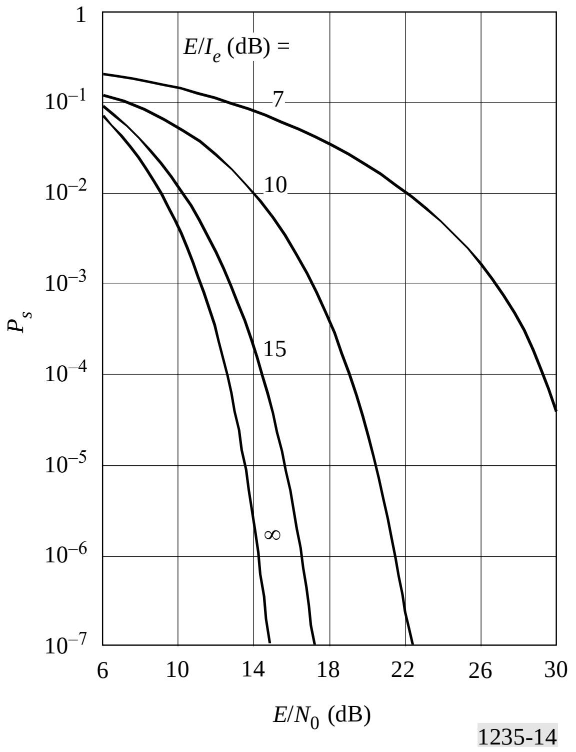

FIGURE 15

P_s versus E/N_0 and E/I_e for 4-ary NCFSK

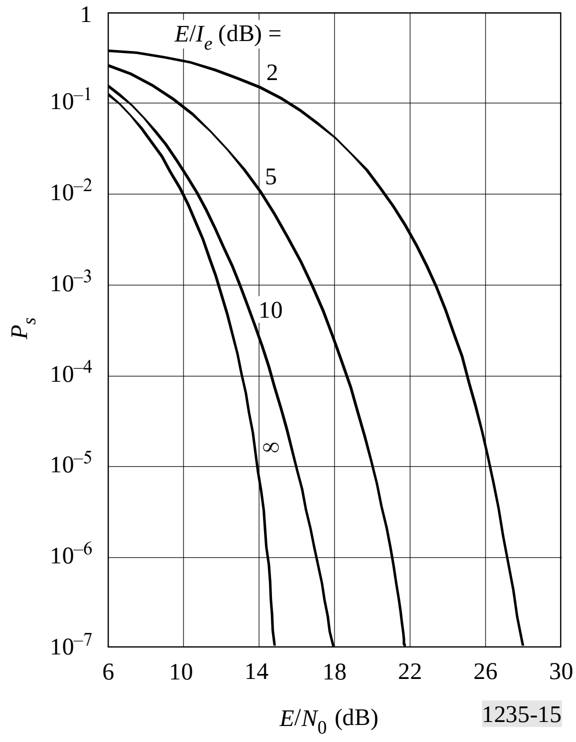

FIGURE 16

P_s versus E/N_0 and E/I_e for 8-ary NCFSK

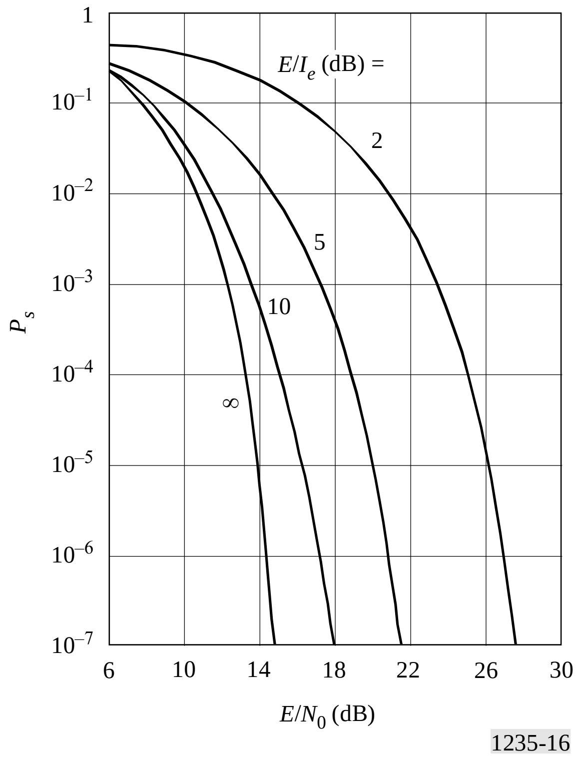

FIGURE 17

P_s versus E_b/N_0 for M-ary bipolar CASK

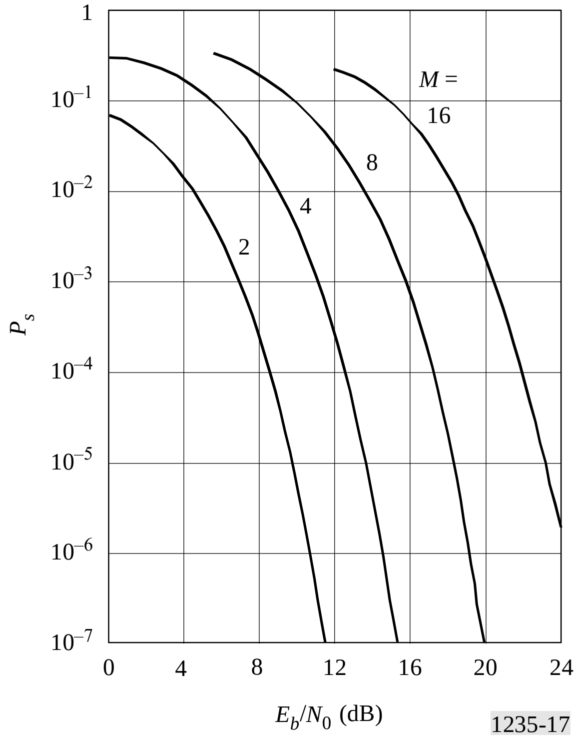

FIGURE 18

P_s versus E/N_0 and E/I_e for 16-ary bipolar CASK

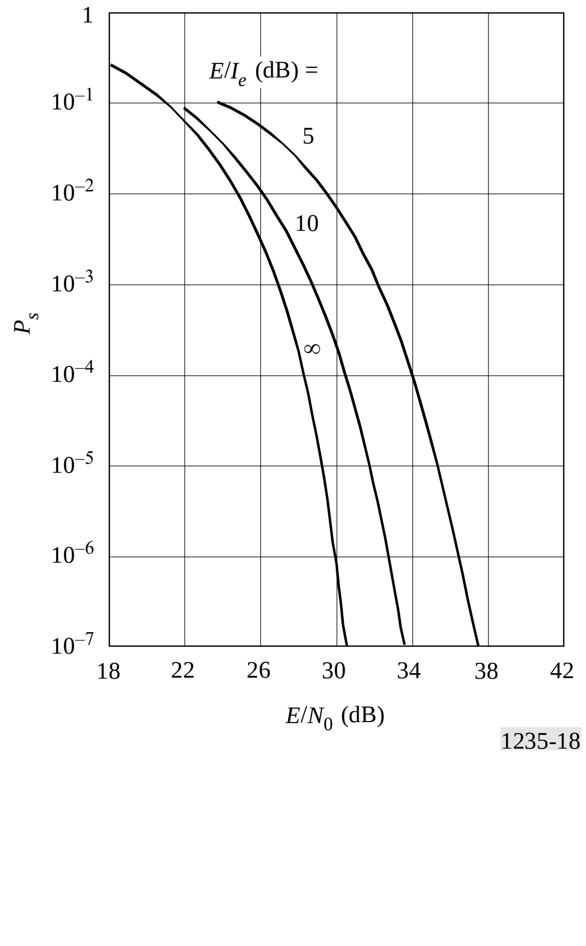

FIGURE 19

P_s versus E_b/N_0 for M-ary unipolar CASK

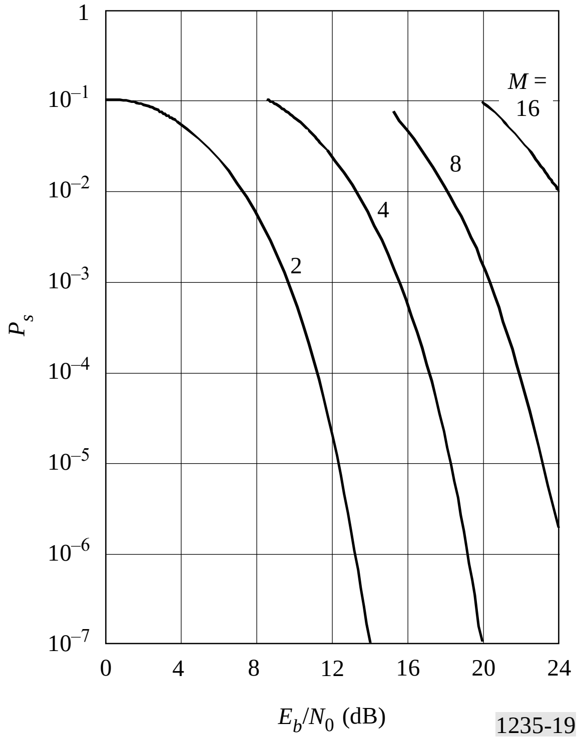

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FIGURE 20

P_s versus E_b/N_0 and E/I_e for binary unipolar CASK (OOK)

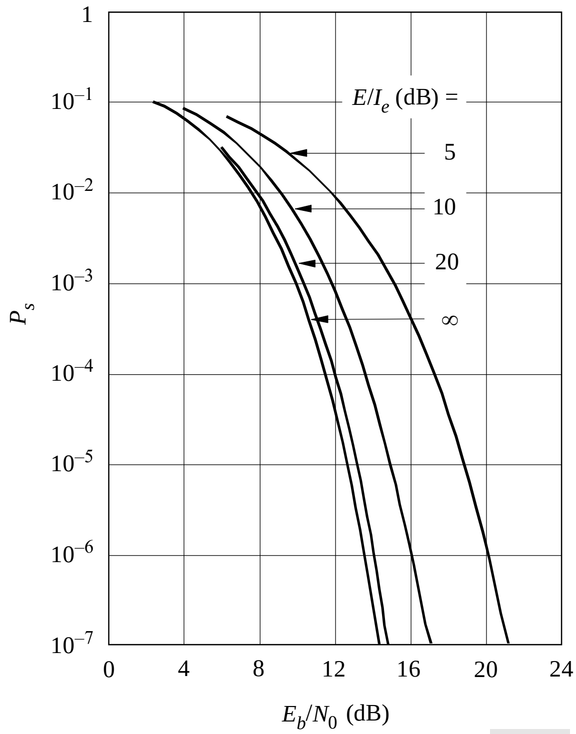

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FIGURE 21

P_s versus E_b/N_0 for M-ary NCASK

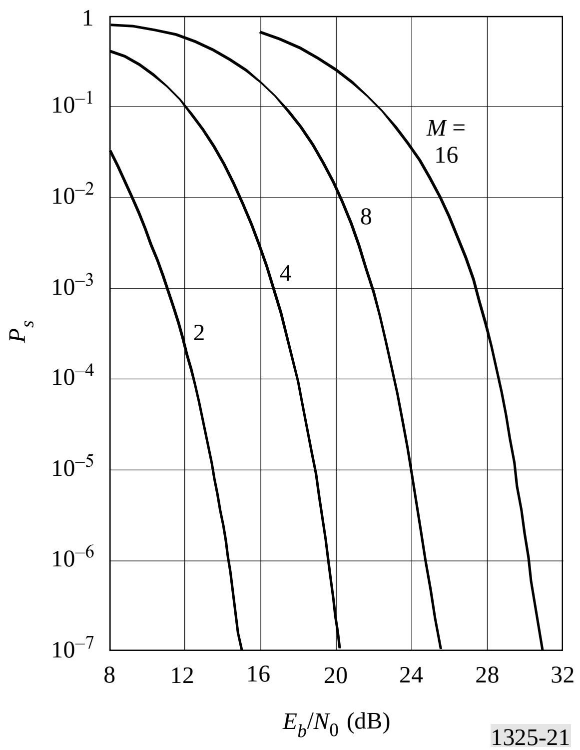


FIGURE 22

P_s versus E_b/N_0 for M-ary QAM and comparison to selected M-ary PSK

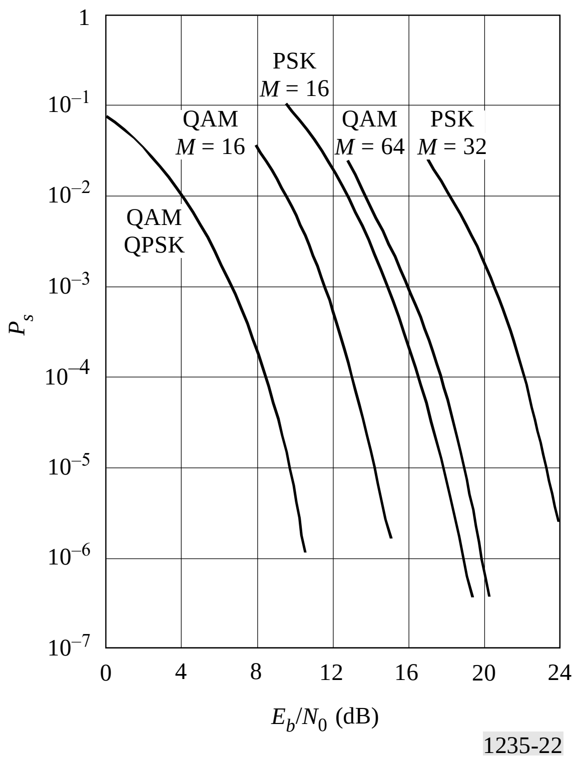


FIGURE 23

P_s versus E/N_0 and E/I_e for 16-ary QAM

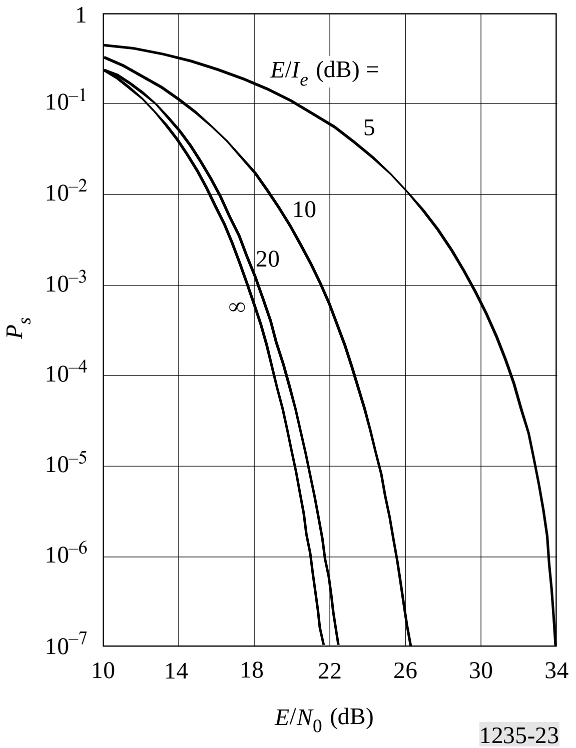
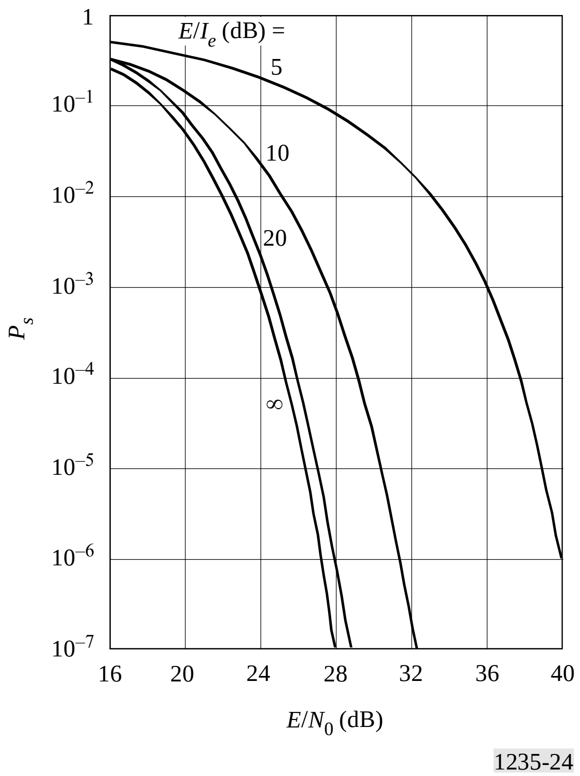


FIGURE 24

P_s versus E/N_0 and E/I_e for 64-ary QAM



ANNEX 2

Performance function for multiple PSK systems with more than one interfering systems

NOTE 1 – All graphs in this Annex with $K = 1$ (a single interfering system) duplicate similar graphs in Annex 1, where S/N and M values are equal.

1 Introduction

Several ITU-R Questions, such as ITU-R 18/1, ITU-R 44/1 and ITU-R 45/1, seek methods and results of communication theory that would increase the efficiency of spectrum use. A case of considerable present – and even greater future – interest to high speed data technology deals with the performance of MPSK systems (coherent M-ary, $M = 2, 3, 4, ...$) in the presence of noise and co-channel interference.

2 Definitions

Assume that each M-ary symbol, same as a binary or non-binary signal element, has duration T, and that the received signal waveform in absence of other input is:

$$s(t) = \sqrt{2S} \cos(\omega_0 t + \theta(t)) \tag{5}$$

where the instantaneous coherent phase $\theta(t)$ is some $2\pi\, m/M$, with m an integer $0 \le m < M$. The signal power is S and the signal energy per symbol is $S\,T$. The received noise $n(t)$ is white Gaussian with one-sided spectral density N_0. To signal and noise we add interference $i(t)$. This interference is co-channel when its centre frequency is also ω_0.

Typical MPSK performance is given in P_s versus S/N plots, with one or more parameters identifying the receiver type, filters in the signal path, signal distortions, interference conditions, and so forth.

The two primary terms are:

P_s: symbol error probability. The bit-error probability, which is also often used, cannot exceed P_e nor can it be less than $P_s/\log_2 M$. The two probabilities are equal for $M = 2$

S/N: $10 \log_{10} (S\,T/N_0)$ is signal symbol energy per noise spectral density (dB) exceeds the bit energy over N_0 (dB) by $10 \log_{10} (\log_2 M)$ dB, and can be interpreted as signal-to-noise power ratio $S/(N_0\,T^{-1})$ (dB).

3 Results

3.1 Theoretical results

The performance of various MPSK systems has been studied by many workers.

Figure 25 shows the performance of a sampler or matched filter receiver in the presence of noise and no interference. The curves are parametric in $M = 2, 4, 8$ and 16.

The inclusion of interference $i(t)$ starts with specification of the number of distinct constant envelope, random angle, interfering signals contained in $i(t)$. Let us denote this multiplicity by K. Thus:

$$i(t) = i_1(t) + i_2(t) + ... + i_k(t)$$

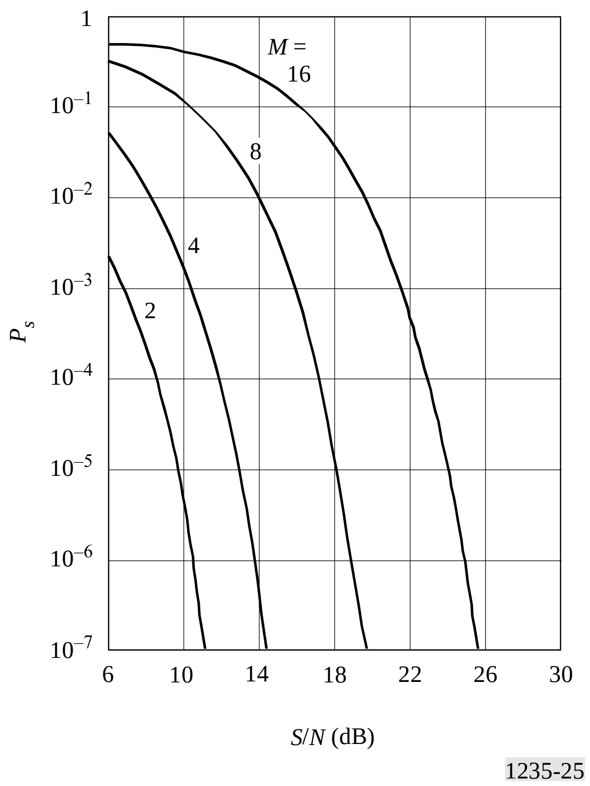
and the total interference power

$$I = I_1 + I_2 + ... + I_k$$

correspondingly. The signal-to-interference ratio (*S*/*I*) is defined as the ratio of desired-to-undesired (interfering) signal power (dB). It is the same as the ratio of signal energy per symbol (or per bit) to interference energy per symbol (or per bit) (dB).

FIGURE 25

Performance of ideal MPSK with no co-channel interference



The simplest case occurs when the interference is a single unmodulated carrier, $K = 1$, at centre frequency ω_0 and with a random uniformly distributed phase. The performance degradation caused by this interference to ideal MPSK is shown in Figs. 26a) to 26d).

The effect of $K > 1$ is to further deteriorate the performance. The effect is shown in Figs. 27a and 27b, each chosen for a fixed *S*/*I* value but both with $M = 2$. It seems that $K = \infty$ should have the worst effect of all choices of *K*.

When the co-channel interference contains modulated constant envelope signals, the effects become far more complex and are not well documented. While theoretical estimates for a single, $K = 1$, angle modulated interference suggest performance degradation shown in Fig. 28, more results can be deduced through simulation. Figure 29 shows the results from a simulated QPSK ($M = 4$) receiver being interfered with by a modulated undesired QPSK signal. The data shows that for high signal-to-interference ratios the theoretical derivation agrees reasonably well with the simulated results. For low signal-to-interference ratios there is considerable difference between the two procedures which is caused by the approximations inherent in the analytic approach. The results in general indicate that the theoretical bounds are valid and for low signal-to-interference ratios additional analytic complexity needs to be considered. In particular, the simulation should be extended to the arbitrary *M* phase case and to a multiplicity *K* of interferers.

3.2 Measured results

The probability of errors for a 4-phase PSK receiver subjected to interference were measured using the test set up shown in Fig. 30. The PSK demodulator included a carrier recovery circuit with a single tuned filter that had a bandwidth of 400 kHz and a detection/decision circuit.

FIGURE 26

Performance of ideal MPSK with a single K = 1 unmodulated interference

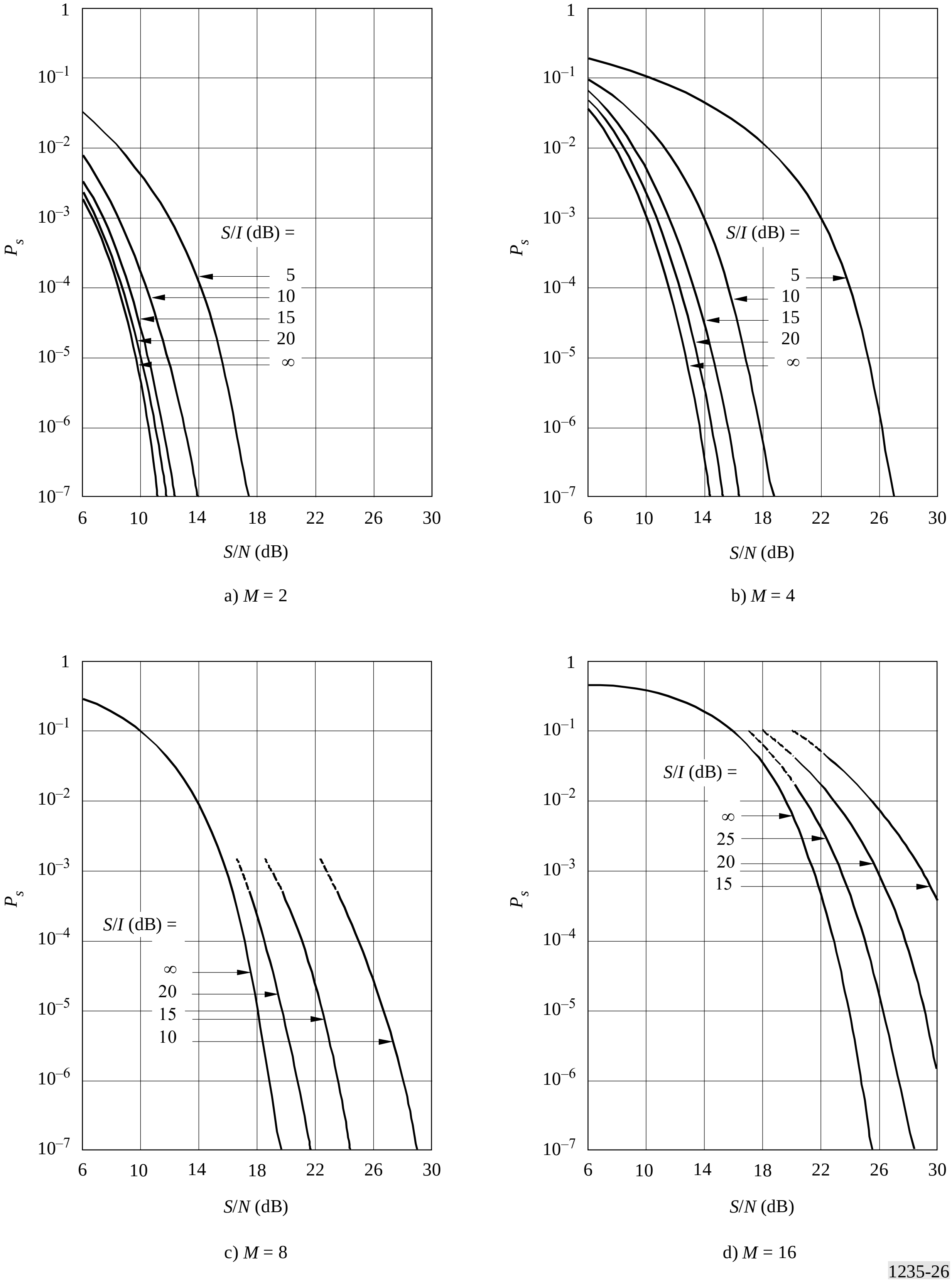
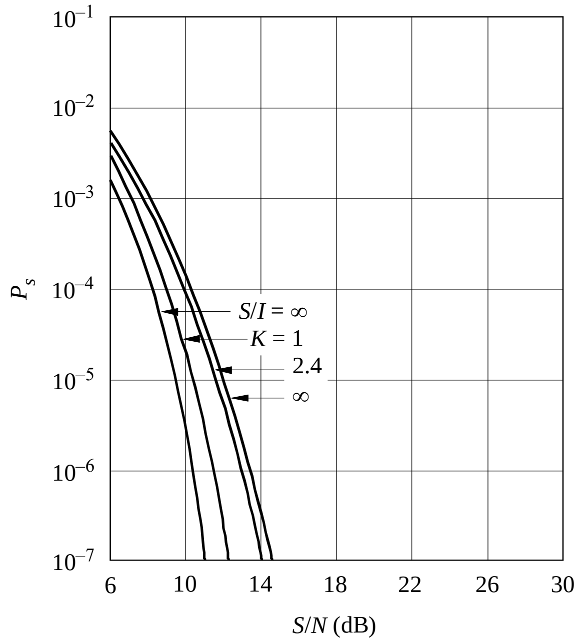


FIGURE 27a

The effect on binary (*M* = 2) MPSK performance by *K* = 1, 2, 4... unmodulated interfering signals



S/*I* = 15 dB
K: number of interfering signals

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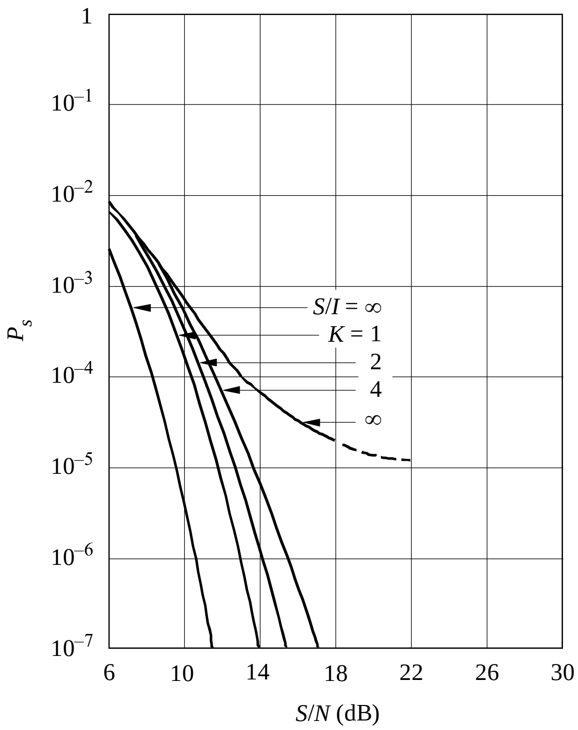
FIGURE 27b

The effect on binary (*M* = 2) MPSK performance by *K* = 1, 2, 4... unmodulated interfering signals



S/*I* = 10 dB
K: number of interfering signals

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FIGURE 28

Binary (M = 2) MPSK performance estimate for a single (K = 1) interference with arbitrary angle modulation

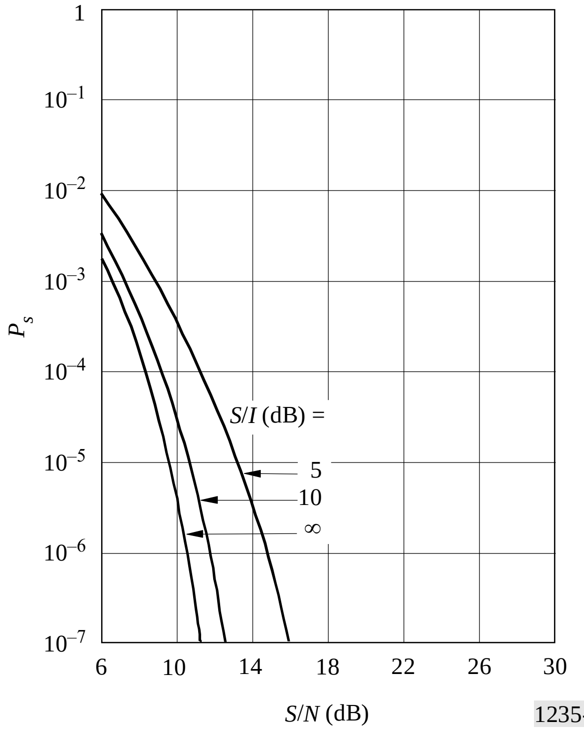

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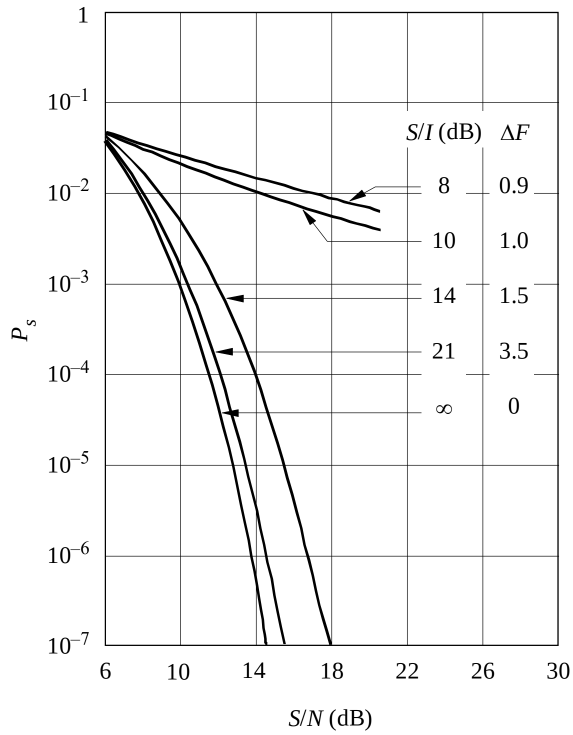
FIGURE 29

QPSK (M = 4) performance simulation results for a single (K = 1) QPSK interference that has a normalized frequency offset (ΔF) CW



$$\Delta F = \frac{f_1 - f_0}{\text{Symbol rate}}$$

1235-29

The desired signal was a 4-phase PSK signal modulated at 30 MBd. The interfering signals (I_1, I_2) are unmodulated sinusoidal waves. In Fig. 31, for $K = \infty$ band limited white noise from noise generator II was used on the interference source. The interfering signal in Fig. 32 was a 4-phase PSK signal modulated at 30 MBd.

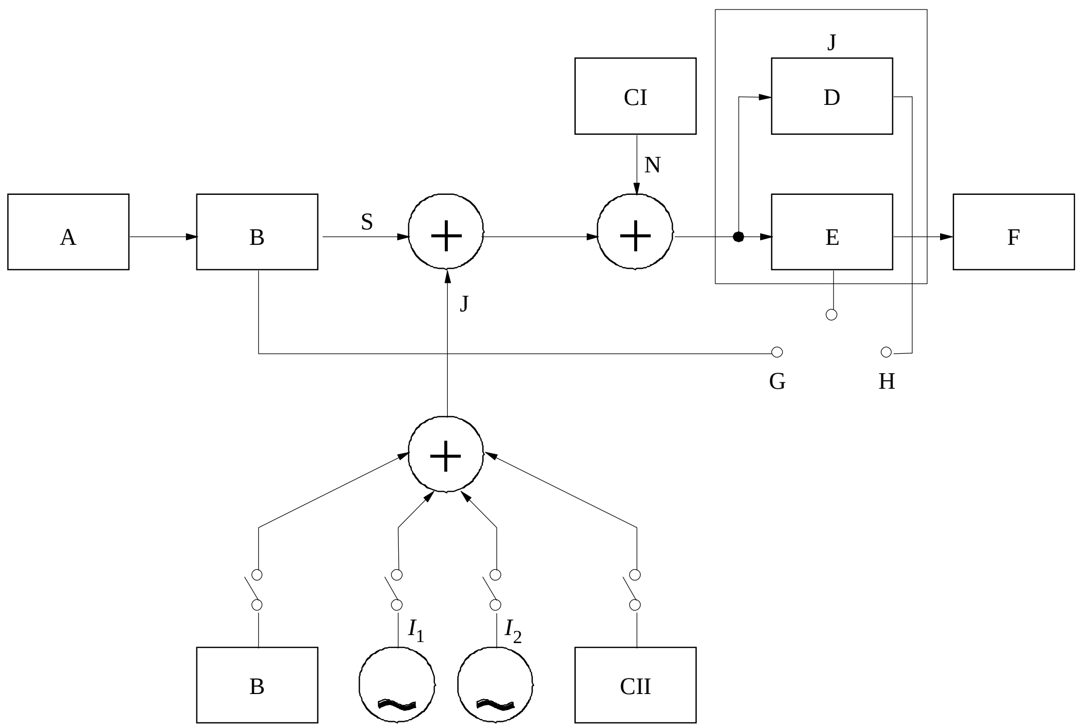
Figure 31 gives measured results on the relation between wanted signal-to-noise power ratio (S/N) and bit error ratio (P_e) with the number of interfering signals (K) as a parameter. The results have the same tendency as the calculated results (Fig. 27b).

Figure 32 shows the measured results on the relation between S/N and P_s with Δf and S/I as parameters. It can be seen from this figure that bit error ratio increases when the value of Δf approaches zero. This seems to result from the effects of interfering signals on the carrier recovery circuit.

Figure 33 shows equivalent change in S/N versus S/I when the desired signal is interfered by 4-phase PSK signal modulated at 30 MBd. The equivalent change in S/N is the difference between S/N required to obtain a given bit error ratio (1×10^{-4} or 1×10^{-6}) in the absence of interference and the S/N required for the same error ratio in the presence of interference. The above results indicate that the effects of the interfering signal on the carrier recovery circuit cannot be neglected in the region of small value of S/I.

FIGURE 30

Simplified block diagram for the measurement

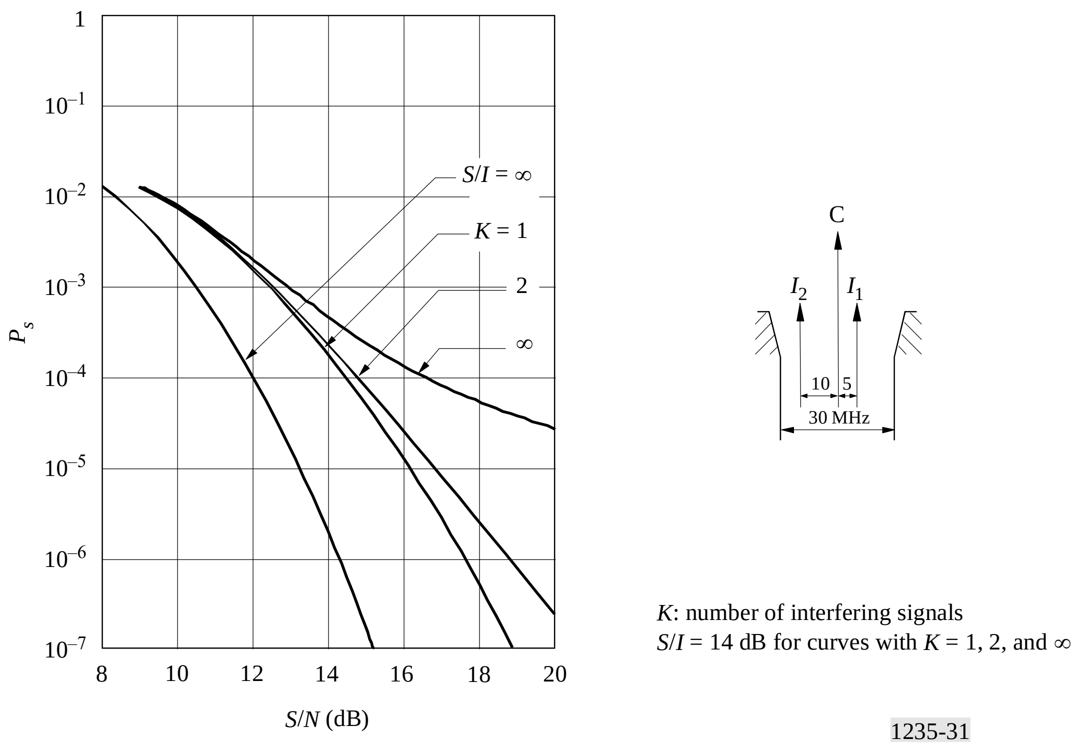


A: PN generator
B: 4-PSK modulator
C: noise generator
D: carrier recovery
E: detection decision
F: P_e measurement
G: original carrier
H: recovered carrier
J: demodulator

1235-30

FIGURE 31

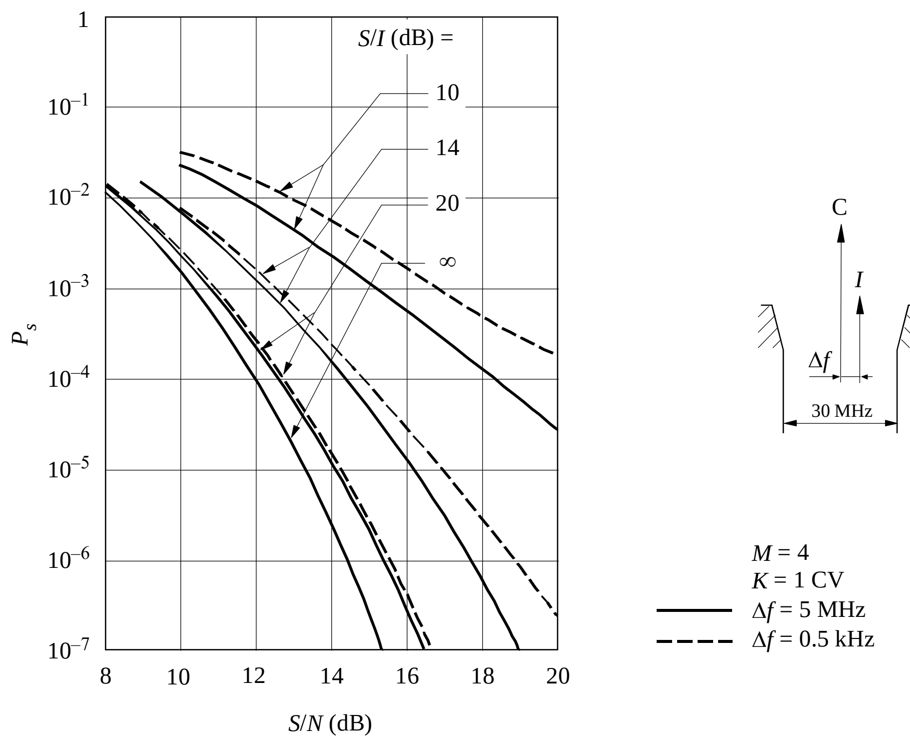
Measurement results on performance of 4-phase PSK with multiple unmodulated interferences



1235-31

FIGURE 32

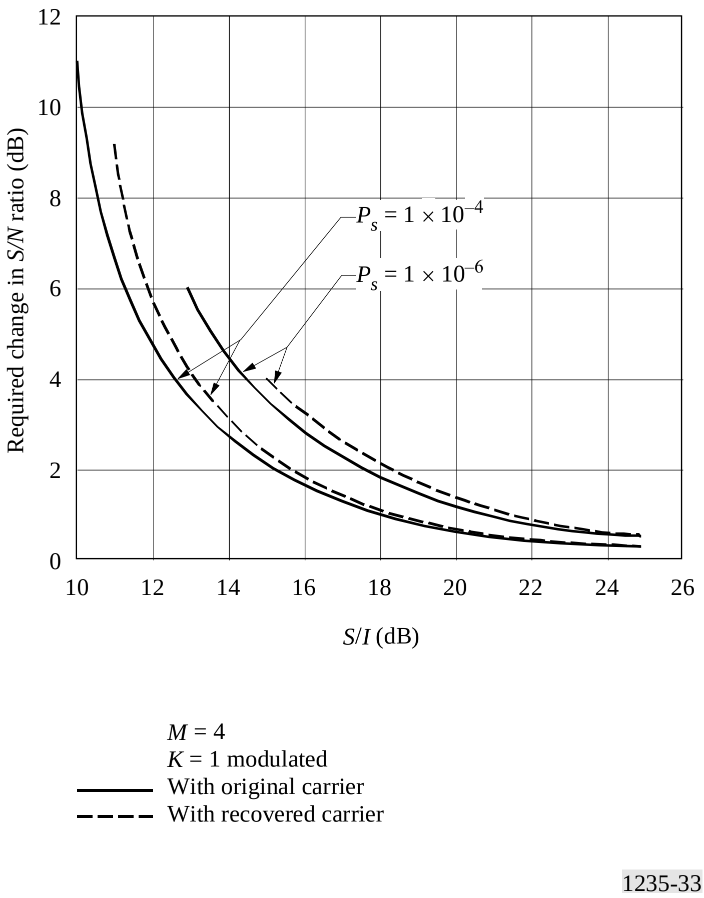
Measurement results on performance of 4-phase PSK with a single unmodulated interference



1235-32

FIGURE 33

Change in *S*/*N* ratio versus *S*/*I* ratio for error rates P_s of 1×10^{-4} and 1×10^{-6}



4 Analytical method

An approximate analytical method can be used to calculate the probability of false symbol reception of *M*-ary PSK signals interfered with by "*K*" numbers of interference signals. The method proceeds from two measurements to the calculation of the expected P_s.

4.1 Measured parameters

With the interfering transmitters turned off or filtered out, the *S*/*N* value of the desired signal is measured (or calculated from site engineering data) at the input of the receiver demodulator, along with "*T*", the desired symbol duration defined for equation (1). Then, with the desired transmitter turned off, the interfering signals are measured as I_j/N at the input to the receiver demodulator, along with the interfering symbol duration (T_j) and frequency offset (Δf_j) from the receiver tuned frequency for each interfering signal.

4.2 Calculation of parameters

In a manner similar to equation (1), the following calculations are made, using the measured data from § 4.1.

$$E/N_0 = (S/N)\,(B\,T) \tag{6a}$$

and:

$$E_{Ij}/N_0 = (I_j/N)\,(B\,T_j) \tag{6b}$$

The following two parameters are then calculated:

$$\rho_0 = 10^{0.1(E/N_0)} \tag{7}$$

and:

$$\rho_{Ij} = 10^{0.1(E/N_0 - E_{Ij}/N_0)} \tag{8}$$

The parameter $h(\Delta f_j)$ is defined in the following way:

$$h\left(\Delta f_j\right) = 10^{-0.05 FDR(\Delta f_j)} \tag{9}$$

where $FDR(\Delta f_j)$ is the frequency dependent rejection as specified in Recommendation ITU-R SM.337 for each Δf_j measured in § 4.1.

The modified Bessel function $I_0(x)$ which is required for later calculations, can be applied with the help of the following formula, where $t = x/3.37$:

$$I_0(x) = 1 + 3.515\,t^2 + 3.090\,t^4 + 1.207\,t^6 + 0.266\,t^8 + 0.036\,t^{10} + 0.005\,t^{12} \qquad \text{for } t \leq 1 \tag{10}$$

or

$$I_0(x) = \frac{\exp(x)}{\sqrt{x}}\left(0.399 + 0.013/t + 0.002/t^2\right) \qquad \text{for } t > 1 \tag{11}$$

These parameters can then be utilized in the formulae of Table 2 to arrive at $\varphi(d_0)$ which is the effective signal-to-noise ratio at the input of the demodulator caused by both interference and thermal noise (dB).

The order of the calculations of Table 2 is as follows:

– Calculate the parameters β_1, β_2 and d_0 from ρ_0, ρ_{Ij}, M and $h(\Delta f_j)$.

– Calculate the parameter $\varphi''(d_0)$, choosing one of the two available formulas.

– Calculate the parameter $\varphi(d_0)$, choosing one of the two available formulas.

– Calculate the function $F(d_0)$, choosing one of the two available formulas and tilizing the modified Bessel function.

4.3 Calculate P_s

Finally, the probability of false symbol reception in the *M*-ary PSK receiver in consideration is calculated by the following approximate expression:

$$P_s = F(d_0)\exp\left[\varphi(d_0)\right] \Big/ \sqrt{2\pi\ \varphi''(d_0)} \tag{12}$$

5 Conclusions

Co-channel interference degradation from more than one interfering system on the error probability performance of coherent MPSK modems under the quoted conditions can be estimated from the given theoretical curves in Figs. 25 to 28, the simulated results given in Fig. 29 or the experimental measured results given in Figs. 31 and 32.

An analytical method for calculating the potential symbol error probability for a *M*-ary PSK receiver in a multi-signal interfering environment has also been presented with those measurements that are required for the calculations. Taken as a whole, the graphical, simulated and analytical approaches presented will suffice to cover wide range of digital modulation systems, with particular attention paid to PSK modulation.

TABLE 2

Formulae for calculation

Formulae for the parameters:

$$\beta_1 = \sqrt{2\rho_0} \sin \frac{\pi}{M} / 1 + \sum_{j=1}^{K} \rho_{Ij} \; h^2(\Delta f_j)$$

$$\beta_2 = \sqrt{2\rho_0} \; \sin \frac{\pi}{M} - \sum_{j=1}^{K} \sqrt{2\rho_{Ij}} \; \left| h(\Delta f_j) \right|$$

$$d_0 = \max\,(\beta_1; \; \beta_2)$$

$$\varphi''(d_0) = 1 + \sum_{j=1}^{K} \rho_{Ij} \; h^2(\Delta f_j) \qquad \text{for } \beta_1 > \beta_2$$

$$\varphi''(d_0) = 1 \qquad \text{for } \beta_1 < \beta_2$$

$$\varphi(d_0) = -d_0 \sqrt{2\rho_0} \; \sin \frac{\pi}{M} + \frac{1}{2} d_0^2 \left(1 + \sum_{j=1}^{K} \rho_{Ij} \; h^2(\Delta f_j) \right) \qquad \text{for } \beta_1 > \beta_2$$

$$\varphi(d_0) = -d_0 \left(\sqrt{2\rho_0} \; \sin \frac{\pi}{M} - \sum_{j=1}^{K} \sqrt{2\rho_{Ij}} \; \left| h(\Delta f_j) \right| \right) + \frac{d_0^2}{2} \qquad \text{for } \beta_1 < \beta_2$$

Formulae for the functions:

$$\mathrm{F}(d_0) = \frac{1}{d_0} \prod_{j=1}^{K} \mathrm{I}_0 \left(d_0 \sqrt{2\rho_{Ij}} \; \left| h(\Delta f_j) \right| \right) \exp\left(-\frac{1}{2} d_0^2 \; h^2(\Delta f_j) \, \rho_{Ij} \right) \qquad \text{for } \beta_1 > \beta_2$$

$$\mathrm{F}(d_0) = \frac{1}{d_0} \prod_{j=1}^{K} \mathrm{I}_0 \left(d_0 \sqrt{2\rho_{Ij}} \; \left| h(\Delta f_j) \right| \right) \exp\left(-d_0 \sqrt{2\rho_{Ij}} \; \left| h(\Delta f_j) \right| \right) \qquad \text{for } \beta_1 < \beta_2$$