REPORT ITU-R SA.2066

Means of calculating low-orbit satellite visibility statistics

(2006)

CONTENTS

1 Introduction

The increasing use of space stations in circular low-orbit in the space research service (and other services) necessitates the development of dynamic sharing models in which the potential interference from the space station can be treated as a time-varying function. This Report defines analytical tools for calculating visibility statistics for low-orbiting spacecraft in circular orbits (see Note 1) as seen from a specific point on the Earth's surface.

NOTE 1 – This Report only deals with circular satellite orbits in which the orbital period is not an even multiple of the Earth's rotational period.

Section 2 of this Report describes the factors affecting the visibility statistics, presents a bounding equation for determining the percentage of time that a low-orbiting satellite will occupy specified regions of the orbit shell visible to an earth station, and contains summary charts giving the maximum duration a low-orbiting satellite spends in certain regions of the orbit shell as a function of several parameters. Section 3 develops the probability density function (pdf) of a satellite occupying specific locations on the orbit shell, illustrates how the pdf may be used to calculate the statistical characteristics of interference to low-orbiting satellites resulting from emissions from stations in the FS, and demonstrates the computation of the pdf of the interference to FS systems assuming the power flux-density (pfd) of the emissions of the low-orbiting satellites conform to a specific profile. Section 4 proposes a simplified method to calculate the visibility statistics of earth stations or terrestrial stations using an antenna with a beam of circular cross section and also presents a manual visibility computation method based on the use of a spreadsheet to calculate the visibility statistics of earth stations or terrestrial stations employing an antenna with a beam of a more complex cross section. Finally, section 5 provides a means to calculate the coordinates in inertial space of the intersection of two orbital planes. This section is particularly useful for predicting the conjunction of satellites in sun-synchronous orbits whose orbital planes are offset.

2 Percentage of time and maximum duration for a low-orbiting spacecraft occupying a defined region

Even for the simplest of dynamic sharing models, at least six specific system parameters must be evaluated to define precisely the primary time dependent statistics of a low-orbit space station as seen from a location on the Earth's surface.

The time dependent statistics are:

– the longest time of passage of a space station through the main beam of a ground antenna (discussed in § 3);
– the long-term percentage of time that the space station spends in various areas of the orbit sphere as seen from the ground station.

The first statistic is important in that it defines the longest continuous duration of noise power into the ground receiving system from the space station. The second set of statistics, after convolution with transmit and receive antenna patterns, and range loss, can be used to develop interference-to-noise (I/N) relations as a function of time for the dynamic sharing model. In one sense then, I/N versus time relations can be treated in a method similar to the signal strength versus time relations derived from atmospheric propagation statistics. However, instead of a receiver experiencing change in the S/N ratio as a statistical function of time, it experiences a change in signal-to-noise-plus-interference ratio, as a statistical function of time, based upon the low-orbit space station model parameters.

The specific parameters which define the long-term visibility statistics of a space station in a low circular inclined orbit as seen from a receiving system on the Earth's surface are:

– altitude of the space station, H (km);

– inclination of the space station orbit, i (degrees);
– latitude of the ground station, La (degrees);
– pointing azimuth of the ground station antenna measured from North, Az (degrees);
– pointing elevation of the ground station antenna measured from the local horizontal plane, El (degrees);
– angular area of the region of interest, δA.

The last parameter may take on several different physical interpretations depending upon the purpose of the analysis. For instance, it may be the angular area of the main beam of the ground station antenna or it may be taken as an angular area expressed by an azimuth "width" of δAz (degrees) and an elevation height expressed as δEl (degrees).

2.1 Bounding equation for percentage of time spacecraft is in defined region

The bounding equation is given below and may be used to determine the percentage of time that a low-orbit spacecraft will reside in certain regions visible to a ground station over long periods of time:

$$T(\%) = \frac{\delta\lambda}{2\pi^2}\left(\sin^{-1}\left[\frac{\sin(L+\Delta L)}{\sin i}\right] - \sin^{-1}\left[\frac{\sin L}{\sin i}\right]\right)\times 100 \tag{1}$$

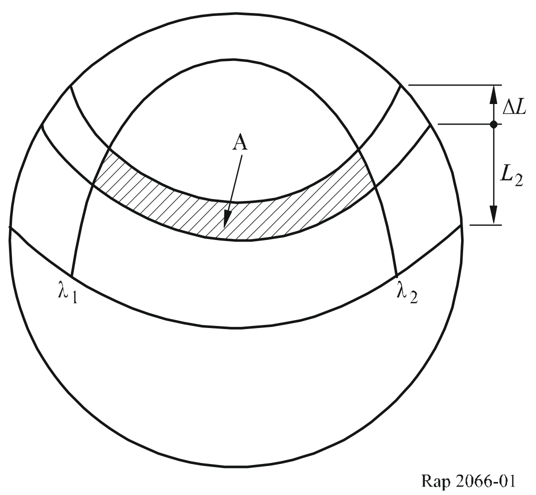
where:

L, ΔL: latitude limits of the region on the orbital shell (see Fig. 1)

$\delta\lambda$: longitudinal extent of the region on the orbital shell, between the longitude limits of λ_1 and λ_2 (as seen in Fig. 1)

i: inclination of the satellite orbit (all angles in rad).

FIGURE 1



2.2 The maximum time a satellite spends in the beam of a ground station

This section provides worst case numerical data on one aspect of frequency sharing with low-orbit, inclined orbit satellites. Such sharing is influenced by the amount of time that an "unwanted" and potentially interfering satellite appears within the 3 dB beamwidth of a ground station. This parameter is evaluated for several orbit altitudes and for two "bounding" elevations of the receiving antenna. The numerical results developed in this paper represent an upper bound on the length of time a spacecraft at a given altitude will appear within the beam of a ground station.

The time a satellite spends in a ground station's beam is a function of the beam's width, the elevation of the beam and the altitude of the satellite. The worst case, i.e. when the satellite spends the maximum possible time in the beam, occurs when the ground station is located at the equator with a beam of elevation = 0° and the satellite is traveling east along an orbit with 0° inclination. The time the satellite spends in the beam depends upon the satellite's velocity relative to the velocity of the beam as it rotates with the Earth, and upon the length of the intersection of the orbit with the beam.

The maximum time that a spacecraft can spend in the main beam of an antenna is shown in Figs. 2 and 3 for antenna elevations of 0° and 90° respectively, and refers to a variety of orbital altitude and beamwidths.

FIGURE 2

Maximum time in beam plotted against beamwidth at elevation of 0°

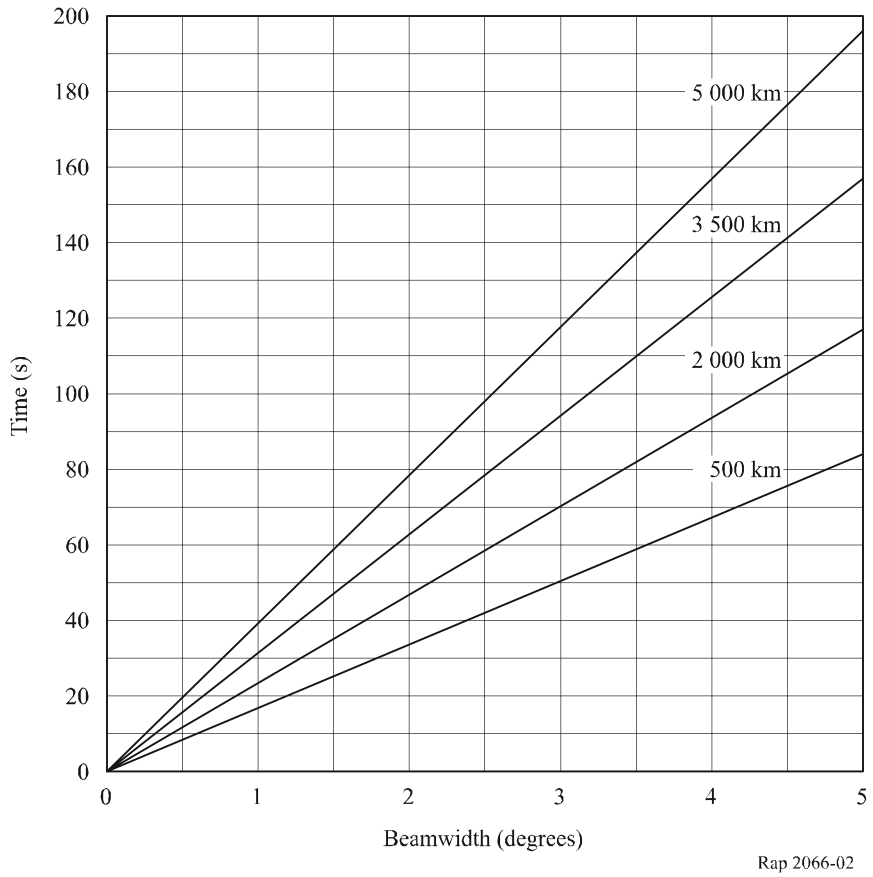
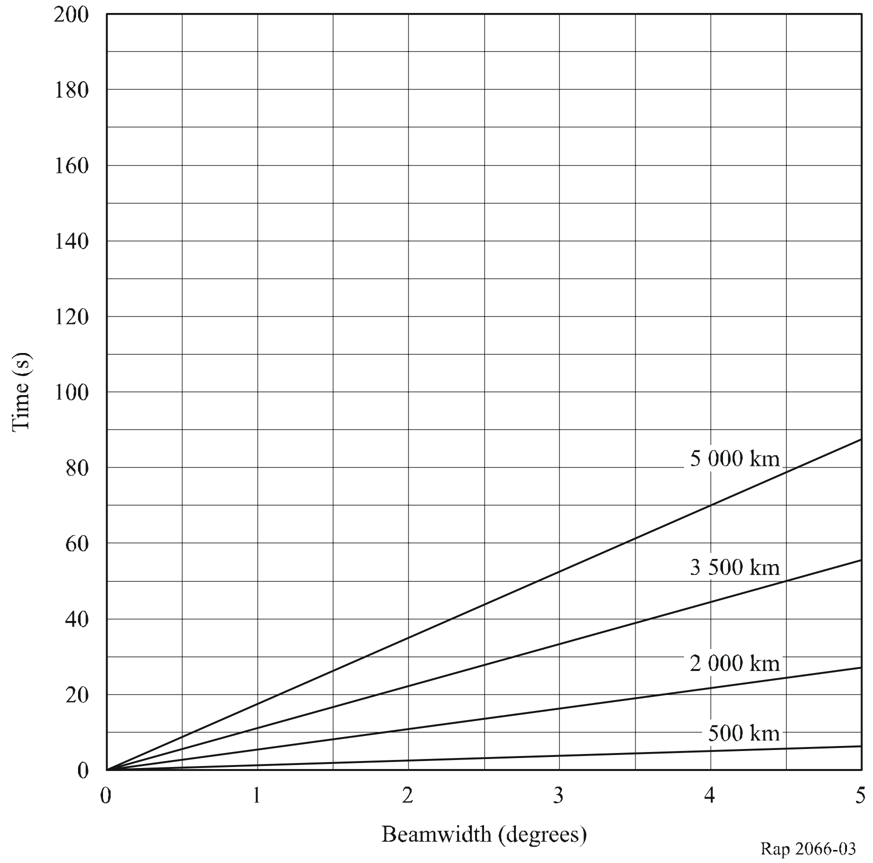


FIGURE 3

Maximum time in beam plotted against beamwidth at elevation of 90°

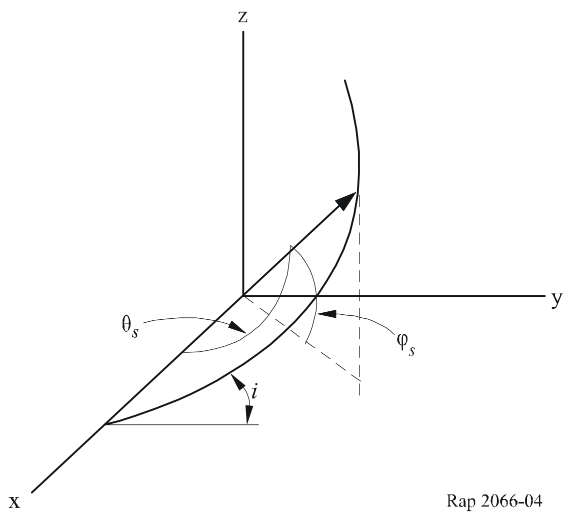


3 Probability density function pdf of the position of a low-orbiting satellite on the orbit shell

The position, i.e. the latitude and longitude of an orbiting satellite on the orbit shell relative to a fixed point on Earth, is a function of two independent parameters: the position of the satellite in its orbit plane; and the longitude of the observation point on Earth relative to the orbit plane. The geometry used for this analysis is shown in Fig. 4. It is assumed that the satellite is in a circular orbit at an altitude, h, the inclination of the orbit plane is i, and the period of rotation of the satellite and of the Earth are not directly related.

FIGURE 4

Simplified geometric model of an Earth-orbiting satellite



The coordinate system shown in Fig. 4 is a right-handed, geocentric system with the x-y plane corresponding to the equatorial plane, and the x-axis pointing in an arbitrary direction in space (usually the First Point of Aries).

For simplicity, assume that the intersection of the orbit plane and the equatorial plane is the x-axis. The latitude φ_s of the position of the satellite in space is given by:

$$\sin \varphi_s = \sin \theta_s \sin i \qquad (2)$$

where θ_s is the central angle between the x-axis and the position vector of the satellite. For satellites in circular orbits, θ_s is a linear function of time t, i.e. $\theta_s = 2\pi t / \tau$, where τ is the period of the orbit. Equation (2) relates the latitude of the satellite as a function of the central angle θ_s and the orbit inclination angle i.

If the central angle of the position vector of a satellite in a circular orbit is sampled at random times, the angle θ_s will be found to be uniformly distributed between 0 and 2π radians. Stated as a probability density function $p(\theta_s)$:

$$p(\theta_s) = \frac{1}{2\pi} \qquad (3)$$

The pdf of the latitude of the position vector of the satellite may be found using a straightforward transformation technique from probability theory. It may be shown for a random variable x with pdf $p(x)$ that undergoes the transformation $y = g(x)$, that the pdf $p(y)$ of the random variable y is given by:

$$p(y) = \frac{p(x_1)}{|g'(x_1)|} + \ldots + \frac{p(x_n)}{|g'(x_n)|} \qquad (4)$$

where:

$$g'(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

and x_1, ... x_n are the real roots of $y = g(x)$.

Applying the procedure described above to equations (2) and (3) yields the pdf of the latitude of the position vector of the satellite in its orbital plane:

$$p(\varphi_s) = \frac{1}{\pi}\frac{\cos\varphi_s}{\sqrt{\sin^2 i - \sin^2 \varphi_s}} \tag{5}$$

Equation (5) represents the function that would be obtained if the latitude of the satellite were randomly sampled a large number of times. Inspection of equation (5) shows that the expression is defined only for real values of $|\varphi_s| \leq i$ as expected. It may also be shown that:

$$\int_{-i}^{i} p(\varphi_s)\mathrm{d}\varphi_s = 1 \tag{6}$$

also as expected.

For the satellite to appear at a specific longitude λ_s on the orbit shell relative to the reference point on the surface of the Earth, the orbit plane must intersect the orbit shell at that longitude. The probability of this occurring is uniformly distributed over 2π radians, i.e.:

$$p(\lambda_s) = \frac{1}{2\pi} \tag{7}$$

Finally, since it has been assumed that the period of the satellite and the rotation of the Earth are not directly related, the pdf of the satellite position is the joint probability of two independent events which is given by the product of the individual pdf_s:

$$p(\varphi_s, \lambda_s) = \frac{1}{2\pi^2}\frac{\cos\varphi_s}{\sqrt{\sin^2 i - \sin^2 \varphi_s}} \tag{8}$$

The probability $P(\Delta\varphi,\Delta\lambda)$ of the satellite occupying the region on the orbit shell bounded by latitude φ_s, $\varphi_s + \Delta\varphi_s$ and longitude $\Delta\lambda_s$ is given by:

$$P(\Delta\varphi, \Delta\lambda) = \frac{1}{2\pi^2}\int_{o}^{\Delta\lambda_s}\int_{\varphi_s}^{\varphi_s+\Delta\varphi_s}\frac{\cos\varphi_s \mathrm{d}\lambda_s \mathrm{d}\varphi_s}{\sqrt{\sin^2 i - \sin^2 \varphi_s}} \tag{9}$$

Carrying out the integration yields:

$$P(\Delta\varphi, \Delta\lambda) = \frac{\Delta\lambda_s}{2\pi^2}\left(\sin^{-1}\left[\frac{\sin(\varphi_s + \Delta\varphi_s)}{\sin i}\right] - \sin^{-1}\left[\frac{\sin\varphi_s}{\sin i}\right]\right) \tag{10}$$

3.1 Probability density function of interference to low-orbiting satellites caused by emissions from FS systems

The pdf of interference to low-orbiting satellites caused by emissions from FS systems is a function of the geometry and the pdf of the satellite position. If the interference can be expressed as a function of the coordinates (latitude and relative longitude) of the visible orbit shell, i.e. $I(\varphi_s, \lambda_s)$, then the pdf of the interference to the low-orbiting satellite $p(I)$ is given by:

$$P(I)dI = \iint_{S} p(\varphi_s, \lambda_s)\mathrm{d}\varphi_s \mathrm{d}\lambda_s \tag{11}$$

where S indicates that the integration is to be performed over the segment of the surface of the orbit sphere that contributes a level of interference between the values of I and $I + \mathrm{d}I$.

The function $I(\varphi_s, \lambda_s)$ is a complex function of a number of parameters that include the location of the FS station, the transmitter power spectral density, the directional characteristics of the transmitting antenna gain, the azimuth and elevation angle of the transmitting antenna, the altitude and orbit inclination angle of the satellite, the range to the satellite, the gain of the satellite receiving antenna in the direction of the interference and the operating frequency. An integral involving a function of this complexity is most readily handled through the use of numerical techniques.

The steps used in the numerical procedure are:

Step 1: define φ_s and λ_s as independent variables over the surface of the visible orbit sphere.

Step 2: define an array $I(n)$ corresponding to the range of interest (maximum to minimum value of interference ($I(\varphi_s, \lambda_s)$) where n corresponds to the number of desired increments (e.g. 0.25 dB increments) (this array will be used to store the differential pdf).

Step 3: evaluate $I(\varphi_s, \lambda_s)$ at specific values of φ and λ (this value will be used to point to a specific element n_0 in the array $I(n)$).

Step 4: calculate $p(\varphi_s, \lambda_s)\, d\varphi_s\, d\lambda_s$ and add to the value stored in $I(n_0)$.

Step 5: increment φ and λ over the surface of the visible orbit sphere.

Step 6: repeat steps 3 to 5.

It is noted that evaluating equation (11) numerically results in the transformation of the integral to a summation.

The geometrical parameters required to evaluate $I(\varphi_s, \lambda_s)$ are obtained by using a geocentric coordinate system similar to the one shown in Fig. 4. The main difference is that the coordinate system rotates at the same rate and direction as the Earth. The x-y plane is the equatorial plane and the z-axis is the rotational axis of the Earth. The position of the FS station is assumed, for simplicity, to lie in the x-z plane. The scale of the coordinate system is normalized to the radius of the Earth. Therefore, any distances computed in this coordinate system must be multiplied by the radius of the Earth (6 378 km) to obtain the correct value. The normalized components of the station position vector ***P*** are given by:

$$\boldsymbol{P} = \begin{vmatrix} \cos\varphi_p \\ 0 \\ \sin\varphi_p \end{vmatrix} \tag{12}$$

where φ_p is the latitude of the FS station.

The direction the FS transmitting antenna is pointed, is given by a unit vector that lies in the plane of the local horizontal and is offset from the direction of North by a specified azimuth angle θ_{az}. The components of the antenna pointing vector $\boldsymbol{U_A}$ are given by:

$$\boldsymbol{U_A} = \begin{vmatrix} \cos\varphi_r \cos\lambda_r \\ \cos\varphi_r \sin\lambda_r \\ \sin\varphi_r \end{vmatrix} \tag{13}$$

where:

$$\varphi_r = \sin^{-1}\left(\cos\varphi_p \cos\theta_{az}\right) \tag{14a}$$

$$\lambda_r = \cos^{-1}\left[\frac{-\sin\varphi_p \cos\theta_{az}}{\sqrt{1-\cos^2\varphi_p \cos^2\theta_{az}}}\right] \tag{14b}$$

The values of φ_s and λ_s defining the limits of the visible surface of the orbit sphere may be readily determined. The limits for φ_s are given by:

$$\varphi_{max} = \varphi_p + \varphi_{lim},\ \varphi_{max} \leq i \qquad \text{otherwise } \varphi_{max} = i \tag{15a}$$

$$\varphi_{min} = \varphi_p - \varphi_{lim},\ \ \varphi_{max} \leq \mathrm{i} \tag{15b}$$

where:

$\varphi_{lim} =$ $\cos^{-1}(1/\beta)$

$\beta =$ $1 + h/r_e$

h: altitude of the satellite

r_e: radius of the Earth.

If $\varphi_{min} < i$, then the low-orbiting satellite is not visible at the FS station.

For an arbitrary value of φ_s between the limits φ_{min} and φ_{max}, the limiting values of the relative longitude λ_{min} and λ_{max} on the visible segment of the orbit sphere are given by:

$$\lambda_{max} = -\lambda_{min} = \cos^{-1}\left[\frac{\cos\varphi_{\mathrm{lim}} - \sin\varphi_p \sin\varphi_s}{\cos\varphi_p \cos\varphi_s}\right] \tag{16}$$

Given values for φ_s and λ_s between the limits derived above, the range to the satellite and the angle between the direction the FS station antenna is pointed and the direction to the satellite is most easily obtained using vector analysis. Specifically, the vector to the satellite $\boldsymbol{R}$ is given by:

$$\boldsymbol{R} = \boldsymbol{S} - \boldsymbol{P} \tag{17}$$

where $\boldsymbol{P}$ is the position vector of the FS station as given by equation (12), and $\boldsymbol{S}$ is the position vector of the sample point of the location of the satellite on the orbit sphere given by:

$$\boldsymbol{S} = \beta\begin{vmatrix}\cos\varphi_s \cos\lambda_s \\ \cos\varphi_s \sin\lambda_s \\ \sin\varphi_s\end{vmatrix} \tag{18}$$

The normalized range to the satellite $|\boldsymbol{R}|$ is given by the square root of the sum of the squares of the components of the range vector given in equation (17). The off-axis angle to the satellite is obtained using the scalar product of the antenna pointing vector $\boldsymbol{U_A}$ (whose normalized components are given by equation (13)) and the range vector $\boldsymbol{R}$. The off-axis angle $\varphi_{off\text{-}axis}$ is given by:

$$\varphi_{off-axis} = \cos^{-1}\left[\frac{\boldsymbol{R}\cdot\boldsymbol{U_A}}{|\boldsymbol{R}|}\right] \tag{19}$$

Recommendation ITU-RF.699 sets forth the reference radiation pattern to be used for FS station transmitting antenna with $D/\lambda_f < 100$ and for antennas with $D/\lambda_f > 100$, where D is the diameter of the antenna and λ_f is the wavelength at the operating frequency. The reference radiation pattern to be used for the receiving system on the low-orbiting satellite will be assumed to be isotropic. Using the above assumptions:

$$I(\varphi_s, \lambda_s) = \frac{P_T G_T(\varphi_{off-axis}) G_R \lambda_f^2}{(4\pi R_s)^2} \tag{20}$$

where:

- P_T: transmitter power (or power spectral density)
- $G_T(\varphi_{off\text{-}axis})$: transmitting antenna gain in the direction of the sample point (φ_s, λ_s)
- G_R: receiving antenna gain at the sample point in the direction of the FS station
- λ_f: wavelength of the operating frequency
- R_s: range (in the same dimensions as λ_f) from the FS station to the sample point (i.e. $|\boldsymbol{R}|\, r_e$).

Using the procedure described earlier in this section, the pdf of the interference to a low-orbiting satellite caused by emissions of an FS station is obtained using equations (11) and (20).

An example case has been evaluated to illustrate the results that are obtainable using the analytical procedure described above. For this case it has been assumed that:

– the FS station is located at 38° N latitude;

– the antenna gain is 50 dBi;

– the azimuth angle of the antenna is 90°;

– the operating frequency is 2 050 MHz;

– the transmitter power spectral density at the input to the antenna is 0 dB(W/1 kHz);

– the satellite is in a circular orbit at an altitude of 800 km;

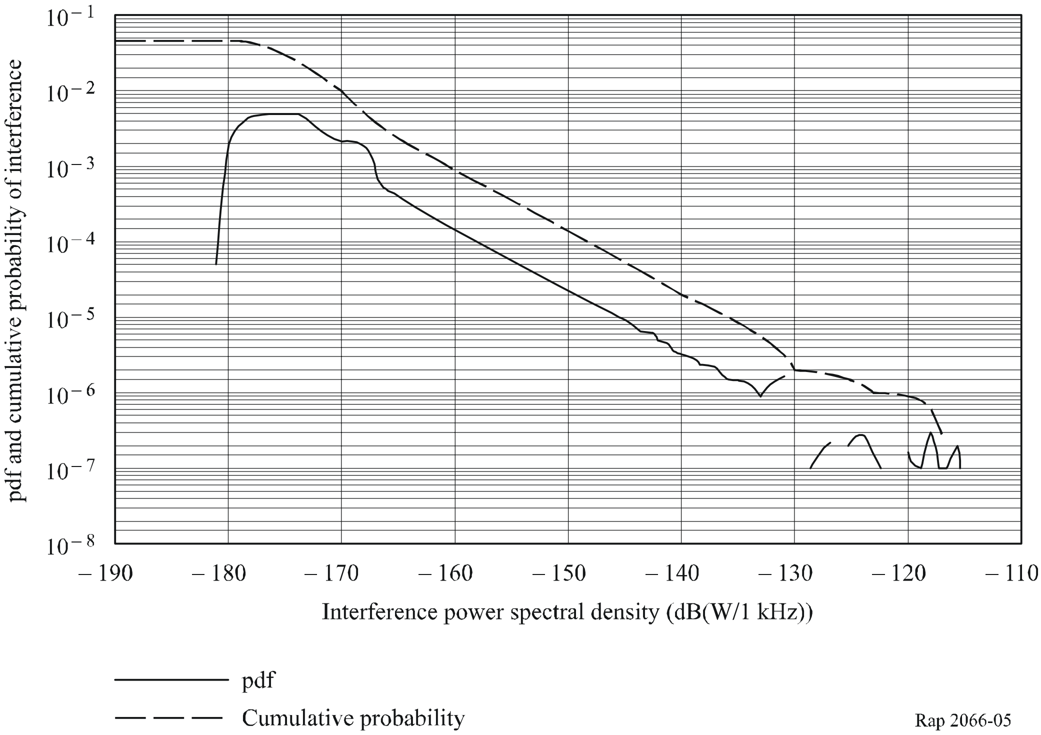
– the inclination of the orbit plane is 90°; and

– the satellite uses an isotropic receiving antenna with a gain of 0 dBi.

The results of the analysis are shown in Fig. 5. The solid curve is the probability density function of interference received by a low-orbiting satellite. The dashed curve gives the cumulative probability that the interference exceeds a specific value. For example, the solid curve shows that the pdf of the interference being on the order of –150 dB(W/1 kHz) is about 2×10^{-5}. Similarly, the dashed curve indicates that the probability of interference exceeding –170 dB(W/1 kHz) is about 1×10^{-2} or 1%.

FIGURE 5

Interference from the example troposcatter station



3.2 Probability density function of interference to FS systems caused by emissions from low-orbiting satellites

The approach used to compute the pdf of interference to FS stations caused by emissions from low-orbiting satellites is a minor extension of the approach described in the previous section. In this case, the incident interference at the FS station is assumed to conform to values of pfd that are specified as a function of the elevation angle at the FS station. The steps of the procedure described in § 3.1 are used. The computation of $I(\varphi_s, \lambda_s)$ becomes:

$$I(\varphi_s, \lambda_s) = \rho(\delta) G_T(\varphi_{off-axis}) \frac{\lambda_f^2}{4\pi} \tag{21}$$

where $\rho(\delta)$ is the spectral pfd, δ is the elevation angle and the other parameters are as defined previously. Equation (19) is used to compute the off-axis angle and the spectral pfd is given by:

$$\rho(\delta) = \begin{cases} -154 & \text{dB(W/(m}^2 \cdot 4\text{ kHz))} \quad \text{for } 0° \le \delta < 5° \\ -154 + 0.5(\delta - 5) & \text{dB(W/(m}^2 \cdot 4\text{ kHz))} \quad \text{for } 5° \le \delta < 25° \\ -144 & \text{dB(W/(m}^2 \cdot 4\text{ kHz))} \quad \text{for } 25° \le \delta < 90° \end{cases} \tag{22}$$

The elevation angle is obtained using the scalar product of the range vector $\boldsymbol{R}$ and the FS station position vector $\boldsymbol{P}$. Noting that cos (90 – δ) = sin δ, then:

$$\delta = \sin^{-1} \frac{[\boldsymbol{R} \cdot \boldsymbol{P}]}{|\boldsymbol{R}|} \tag{23}$$

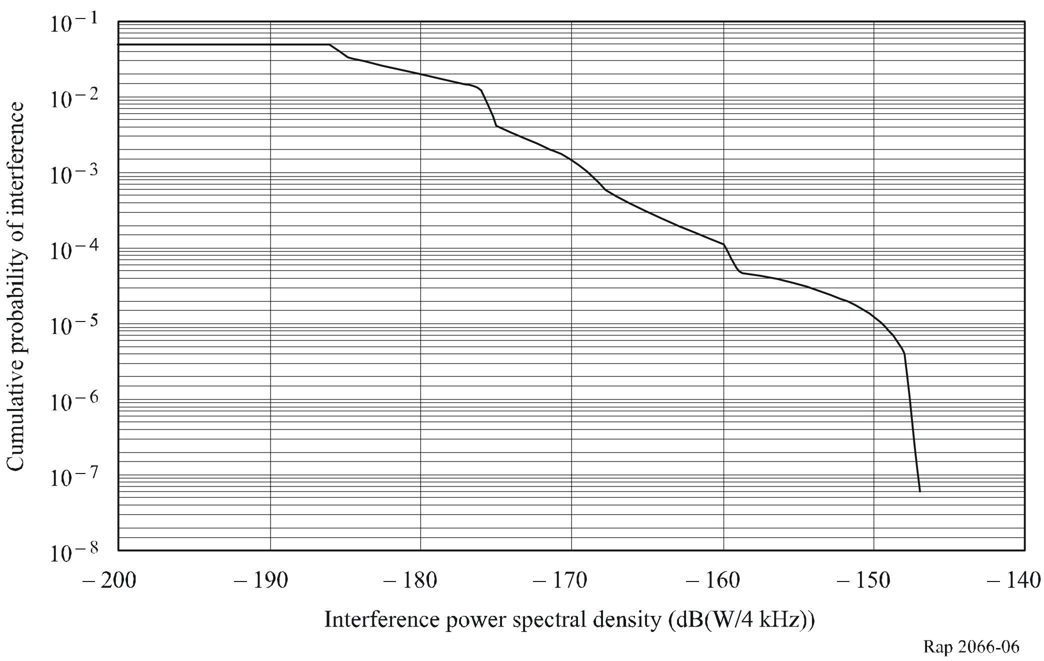
An example has been evaluated for this interference case. Here it has been assumed that:

– the FS station is located at 38° N latitude;
– the receiving antenna gain is 35 dBi;
– the azimuth angle of the antenna is 90°;
– the operating frequency is 2 250 MHz;
– the incident spectral pfd at the FS station is given by equation (22);
– the satellite is in a circular orbit at an altitude of 800 km; and
– the inclination of the orbit plane is 90°.

The results of the analysis are shown in Fig. 6. The solid curve is the cumulative probability that the interference exceeds a particular value. Figure 6 shows that the probability of the interference exceeding –167 dB(W/4 kHz) is on the order of 4×10^{-4}.

FIGURE 6

Interference to example radio-relay station



4 Simplified methods for calculating visibility statistics

The exact method to calculate visibility statistics for a satellite in a circular orbit, and whose orbital period is incommensurate with the rotational period of the Earth, may be determined with the aid of equation (8). This equation, which gives the pdf of a satellite occupying a location at a specified latitude φ_s and longitude λ_s on the orbital shell is repeated as equation (24):

$$P(\varphi_s, \lambda_s) = \frac{1}{2\pi^2} \frac{\cos \varphi_s}{\sqrt{\sin^2 i - \sin^2 \varphi_s}} \tag{24}$$

where:

$p(\varphi_s, \lambda_s)$: pdf

φ_s: geocentric latitude on the orbital shell of interest

λ_s: relative geocentric longitude on the orbital shell of interest

i: inclination of the orbital plane with respect to the equatorial plane.

The probability that a satellite is within a bounded area of the orbital shell, for example, and is "visible" within the 3 dB beamwidth of a receiving antenna, is given by a surface integral:

$$\boldsymbol{P}(\varphi_s, \lambda_s) = \frac{1}{2\pi^2} \oiint_s \frac{\cos\varphi_s}{\sqrt{\sin^2 \boldsymbol{i} - \sin^2 \varphi_s}} \mathrm{d}\varphi_s \mathrm{d}\lambda_s \tag{25}$$

The general solution of equation (25) for an arbitrarily defined area on the orbital shell is difficult. However, for the practical case of a circular antenna beam, certain assumptions lead to a simplified solution. This case is considered in § 4.1.

For a second practical case, where the antenna beam is either circular, or perhaps of a somewhat more complicated shape, a numerical method is described in § 4.2.

4.1 Simplified method for circular antenna beams

Two simplifying assumptions may be made to equation (25) to obtain an accurate estimate of the probability that a satellite will be "visible." The practical case is an earth station or a terrestrial station that uses a relatively high gain antenna with a circular beam pointed at a fixed azimuth and elevation angle. The first simplifying assumption involves the denominator of the integrand in equation (25). If the variation of the value of the denominator is small over the range of the latitude of interest on the orbital shell, then the following simplification may be made

$$\boldsymbol{P}(\varphi_s, \lambda_s) = \frac{1}{2\pi^2} \frac{1}{\sqrt{\sin^2 \boldsymbol{i} - \sin^2 \Phi_s}} \oiint_s \cos\varphi_s \mathrm{d}\varphi_s \mathrm{d}\lambda_s \tag{26}$$

where: $1/\sqrt{\sin^2 i - \sin^2 \Phi_s}$ represents a weighting factor evaluated for Φ_S to be applied to the surface integral. (As will be shown later, Φ_S is taken as the latitude of the centre of the region of interest.) The integrand is greatly simplified by these assumptions since it becomes simply the enclosed surface area A_S on a unit sphere, and the probability reduces to:

$$P(\varphi_s, \lambda_s) = \frac{1}{2\pi^2} \frac{A_S}{\sqrt{\sin^2 i - \sin^2 \Phi_s}} \tag{27}$$

The basic geometry problem to be solved is to determine, A_S, the area of the intersection of a cone (the circular antenna beam) and a sphere (orbital shell). This is facilitated with the second set of assumptions. When the angular dimension of the cone is sufficiently small, the problem becomes the intersection of a cone and a plane that is normal to the sphere at the center of the intersection. It is well known that the intersection results in an ellipse, which for a unit sphere, encompasses an area:

$$A_S = \pi\theta_a\theta_b \tag{28}$$

where:

θ_a: semimajor axis of the ellipse

θ_b: semiminor axis of the ellipse, both angles being measured in rad.

The area A_S may be calculated with the aid of Fig. 7. Figure 7 shows an earth/terrestrial station at point P on the x-axis of a 3-dimensional, coordinate system. The boresight of the station antenna is pointed towards point P_s in the x-y plane at an elevation angle of δ_0. R_s is the range from the station to P_s. The geocentric angle between the station position vector **P** and P_s is θ_0. The major axis of the ellipse, lies in the x-y plane and the minor axis of the ellipse lies in a plane normal to the x-y plane.

The major axis of the ellipse may be determined using a simple relationship between the elevation angle and the central angle:

$$\theta = \cos^{-1}\left(\frac{\cos\delta}{\beta}\right) - \delta \qquad (29)$$

where:

- θ: central angle
- δ: elevation angle
- $\beta =$ $1+h/r_e$;
- h: altitude of the satellite of interest
- r_e: radius of the Earth.

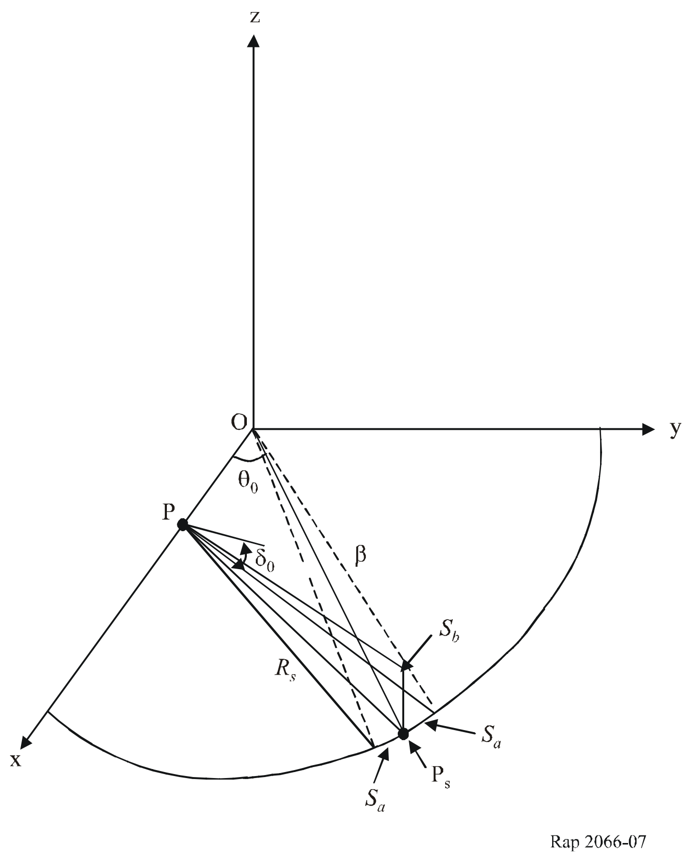
From equation (29), it may readily be shown that θ_a, the semimajor axis of the ellipse on the unit sphere is:

$$\theta_a = \frac{1}{2}\left[\cos^{-1}\left(\frac{\cos(\delta_0 - \varphi_3/2)}{\beta}\right) - \cos^{-1}\left(\frac{\cos(\delta_0 + \varphi_3/2)}{\beta}\right) + \varphi_3\right] \qquad (30)$$

where φ_3 = the beamwidth of interest of the antenna (usually the 3 dB beamwidth); and, the other parameters are as previously defined.

FIGURE 7

Geometry to determine the semimajor and semiminor axes of the ellipse resulting from the intersection of the circular beam and the orbital shell



Referring to Fig. 7, the semiminor axis of the ellipse is determined by first computing the arc S_b, which lies in the plane normal to the x-y plane. The second step is to determine the central angle corresponding to the arc S_b. The central angle is the semiminor axis of the ellipse on a unit sphere. Thus,

$$\frac{R_S}{r_e} = \sqrt{\beta^2 - \cos^2 \delta_0} - \sin \delta_0 \tag{31a}$$

$$\frac{S_b}{r_e} = \frac{R_S}{r_e} \frac{\varphi_3}{2} \tag{31b}$$

but,

$$\frac{S_b}{r_e} = \beta \theta_b \tag{31c}$$

$$\theta_b = \frac{\varphi_3}{2} \frac{1}{\beta} \left[\sqrt{\beta^2 - \cos^2 \delta_0} - \sin \delta_0 \right] \tag{31d}$$

The value for A_S is determined from equations (28), (30) and (31d). Note that θ_a, θ_b and φ_3 must be expressed in rad:

$$A_s = \frac{\pi}{4} \frac{\varphi_3}{\beta} \left[\cos^{-1}\left(\frac{\cos(\delta_0 - \varphi_3 / 2)}{\beta} \right) - \cos^{-1}\left(\frac{\cos(\delta_0 + \varphi_3 / 2)}{\beta} \right) + \varphi_3 \right] \times \left[\sqrt{\beta^2 - \cos^2 \delta_0} - \sin \delta_0 \right] \tag{32}$$

The latitude of the intercept of the boresight of the antenna of the earth/terrestrial station Φ_S is determined in the following way. The geometry is shown in Fig. 8. The station of interest is now located in the x-z plane of a geocentric coordinate system at a latitude of φ_p. The antenna pointing angles are given in terms of the azimuth angle θ_{az} in the clockwise direction from North and the elevation angle δ_0 relative to the local horizontal plane. Shown in Fig. 8 is an oblique spherical triangle with sides a, b and c which are opposite the angles α, θ_{az} and γ. The parameters of the oblique spherical triangle are related to the physical parameters by:

$$b = \pi/2 - \Phi_s \tag{33a}$$

$$c = \pi/2 - \varphi_P \tag{33b}$$

$$a = \cos^{-1}(\beta^{-1} \cos \delta_0) - \delta_0 \tag{33c}$$

The latitude at which the boresight of the antenna intersects the unit sphere and the relative longitude of the point of intersection are given by the Law of Cosines for Sides of an oblique spherical triangle:

$$\sin \Phi_S = \sin \varphi_P \cos a + \cos \varphi_P \sin a \cos \theta_{az} \tag{34a}$$

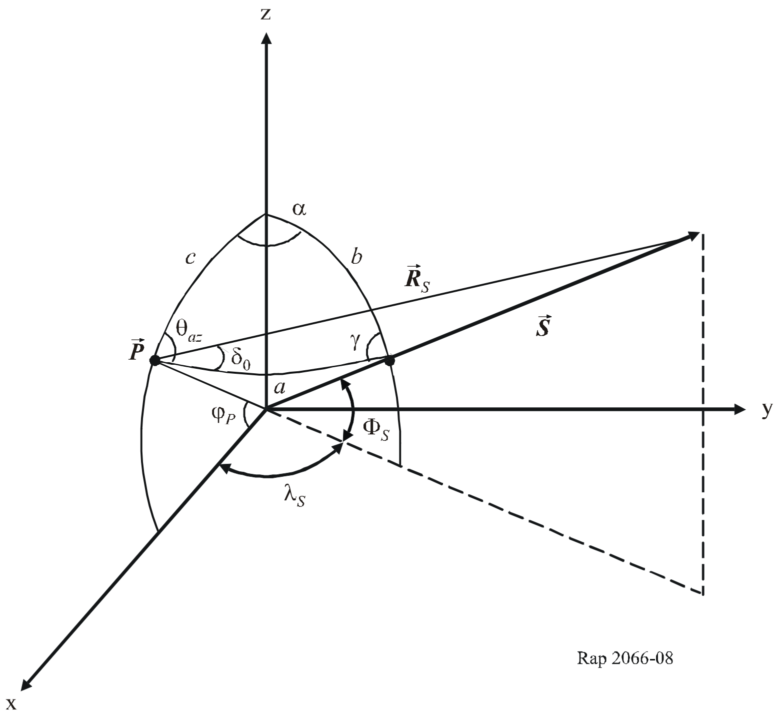
$$\cos \alpha = \frac{\cos a - \sin \Phi_S \sin \varphi_P}{\cos \Phi_S \cos \varphi_P} \tag{34b}$$

Note that λ_S is the angle between the two planes normal to the x-y plane and which contain the arcs b and c. With this observation, λ_S is obtained from the law of cosines when $\Phi_S = \varphi_P = 0$:

$$\lambda_S = \alpha \tag{35}$$

FIGURE 8

Geometry to determine the latitude and longitude of the intercept of the station antenna, given the azimuth and elevation angles, and, the relative altitude of the satellite orbit



4.2 Manual method to calculate visibility statistics

Equation (9) shows that the probability of a satellite occupying a small region on the orbital shell bounded by latitude $\varphi_s - \Delta\varphi_s/2$, $\varphi_s + \Delta\varphi_s/2$ and longitude $\Delta\lambda_s$ is given by:

$$\boldsymbol{P}(\Delta\varphi, \Delta\lambda) = \frac{1}{2\pi^2} \int_0^{\Delta\lambda_s} \int_{\varphi_s - \Delta\varphi_s/2}^{\varphi_s + \Delta\varphi_s/2} \frac{\cos\varphi_s}{\sqrt{\sin^2 \boldsymbol{i} - \sin^2 \varphi_s}} \, d\varphi_s d\lambda_s \qquad (36)$$

Further, as shown by equation (10), carrying out the integration over the area yields:

$$P(\Delta\varphi, \Delta\lambda) = \frac{\Delta\lambda_s}{2\pi^2}\left(\sin^{-1}\left[\frac{\sin(\varphi_s + \Delta\varphi_s/2)}{\sin i}\right] - \sin^{-1}\left[\frac{\sin(\varphi_s - \Delta\varphi_s/2)}{\sin i}\right]\right) \qquad (37)$$

This suggests that the probability over a larger, and perhaps more complicated area could be evaluated by the series:

$$P(\varphi, \lambda) = \sum_{j,k} \frac{\Delta\lambda_j}{2\pi^2}\left(\sin^{-1}\left[\frac{\sin(\varphi_k + \Delta\varphi_0/2)}{\sin i}\right] - \sin^{-1}\left[\frac{\sin(\varphi_k - \Delta\varphi_0/2)}{\sin i}\right]\right) \qquad (38)$$

where φ_k are strips in latitude of height $\Delta\varphi_0$ and of longitudinal extent $\Delta\lambda_j$ such that they are enclosed within the boundary on the orbital shell by the intersection of the antenna beam of interest and the orbital shell. The implementation of this technique is best explained using an example.

Figure 9 shows a typical intersection of an earth station circular antenna beam with the orbital shell. The parameters for this example are given in Table 1. The solution of equation (38) is implemented using a spreadsheet. A square array consisting of 41 × 41 cells is set up that represents the longitude in the x direction and latitude in the y direction on the orbital shell. The latitude and longitude of the center of the array corresponds to the latitude and longitude of the intersection of the antenna boresight with the orbital shell. It also represents the latitude and longitude of the position vector of the satellite when the satellite is aligned with the boresight of the antenna. Thus, each of the other cells in the array represents the possible latitude and longitude of the position vector of the satellite. It remains to determine which of those cells are within the area encompassed by the earth station antenna beam of interest φ_3. This is accomplished with the aid of vectors as shown in Fig. 10.

Figure 10 shows the position vector of the earth station or terrestrial station $\vec{\boldsymbol{P}}$, the position vector of the satellite $\vec{\boldsymbol{S}}$ at an arbitrary location, and the range vector $\vec{\boldsymbol{R}}_{\boldsymbol{S}}$. Since the position vectors of the station and the satellite are either known or assumed, the range vector is determined from:

$$\vec{\boldsymbol{R}}_{\boldsymbol{S}} = \vec{\boldsymbol{S}} - \vec{\boldsymbol{P}} \tag{39}$$

The station and satellite position vectors are given by:

$$\vec{\boldsymbol{P}} = \begin{vmatrix} \cos\varphi_{\boldsymbol{P}} \\ 0 \\ \sin\gamma_{\boldsymbol{P}} \end{vmatrix} \tag{40a}$$

$$\vec{\boldsymbol{S}} = \beta \begin{vmatrix} \cos\varphi_{\boldsymbol{s}} \cos\lambda_{\boldsymbol{S}} \\ \cos\varphi_{\boldsymbol{s}} \sin\lambda_{\boldsymbol{S}} \\ \sin\varphi_{\boldsymbol{s}} \end{vmatrix} \tag{40b}$$

The key to determining if a particular satellite location lies within an area circumscribed by the station antenna beam is the scalar product of the boresight range vector and the range vector associated with the assumed satellite position vector. The angular offset between the two vectors is:

$$\varphi_{j,k} = \cos^{-1}\left[\frac{1}{\left|\vec{R}_{S0}\right|\left|\vec{R}_{j,k}\right|}\vec{R}_{S0}\cdot\vec{R}_{j,k}\right] \tag{41}$$

where:

$\varphi_{j,k}$: off axis angle for the *j*-th value of $\Delta\lambda$ and the *k*-th value of φ

$\vec{\boldsymbol{R}}_{\boldsymbol{S}0}$: boresight range vector

$\vec{\boldsymbol{R}}_{\boldsymbol{j},\boldsymbol{k}}$: range vector for the *j*-th value of $\Delta\lambda$ and the *k*-th value of φ.

For a circular antenna beam, if $\varphi_{j,k} \le \varphi_3/2$, then the satellite will appear within the beamwidth of interest. If this condition is met for the particular cell, then the value in the cell is set to 1, if the condition is not met, the value of the cell is set to 0. Thus, for each row of the 41 × 41 cell array, it is only necessary to sum the number of 1_s in a row and to multiply that number by the factor given in equation (38). Summing the values thus obtained for each row over the 41 rows of latitude yields the estimate for the probability that a satellite will appear within the specified beamwidth of the station antenna.

The step size in both latitude and longitude are parameters that are manually entered into the spreadsheet. Values are selected to ensure that the resulting area, as shown in Fig. 9, is fully contained within the array. In other words, the cells at the extremes in latitude and longitude all contain 0s, and resulting area is sufficiently large to ensure the accuracy of the numerical solution.

TABLE 1

Example parameters and results

Earth station latitude = 40°	Latitude of intersection (Φ_S) = 37.78°
Earth station longitude = 0°	Longitude of intersection (λ_S) = 8.88°
Antenna azimuth angle = 105°	Latitude step size = 0.032°
Antenna elevation angle = 22°	Longitude step size = 0.065°
Satellite altitude = 400 km	Probability of "visibility" = 0.00464%
Satellite inclination = 51.6°	
Beamwidth of interest = 7°	

FIGURE 9

Latitude and longitude of the intersection of a circular beam with the orbital shell (see Table 1)

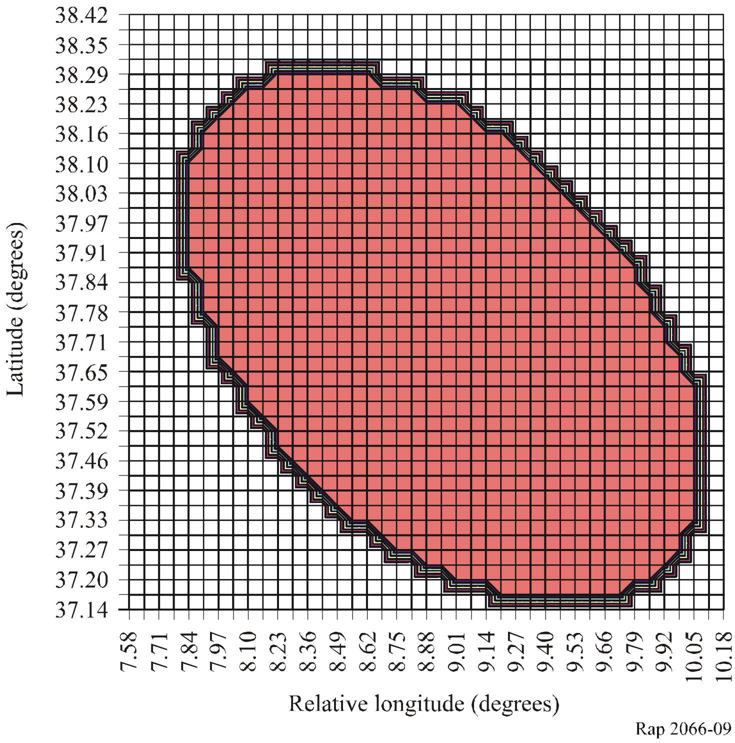


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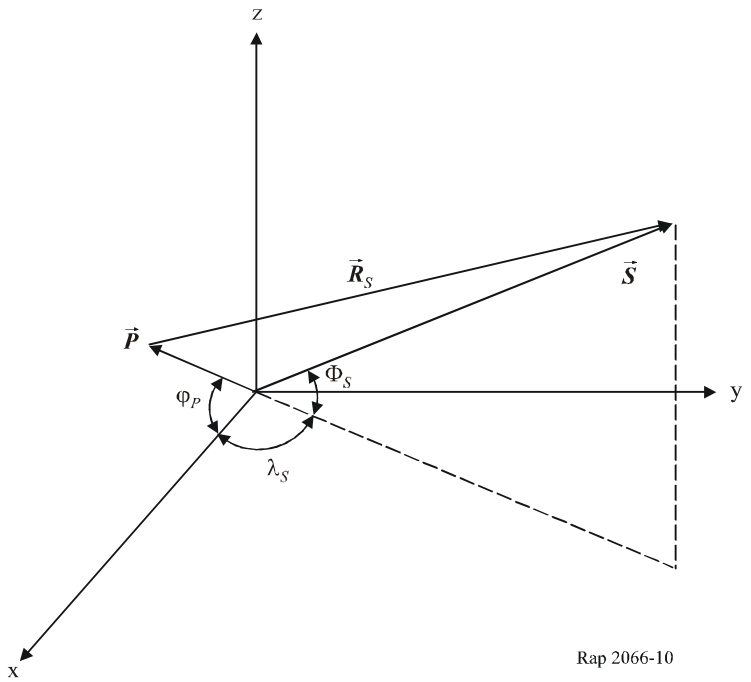
FIGURE 10

Determining the latitude and longitude on the orbital shell which are encompassed by the earth station antenna beam of interest



4.3 Comparison of the numerical results obtained using the simplified and manual methods for circular antenna beams

Table 2 shows representative results for six example cases. In each case it was assumed that the satellite of interest was in an 800 km orbit, inclined by 82° with respect to the equatorial plane. It was further assumed that the orbital period and the precession of the nodes were incommensurate with the rotation rate of the Earth. As a consequence, in a series of trials, the location of the satellite on the orbital shell is a random event.

As Table 2 shows, the error between the results obtained using the two methods was less than 0.4% for the six cases.

TABLE 2

Comparison of the results obtained for the probability of "visibility" for six example cases using the simplified method and the manual method of calculation

	Station location		Antenna data			Probability of visibility		
Case	**Latitude (degrees)**	**Longitude (degrees)**	**Azimuth angle (degrees)**	**Elevation angle (degrees)**	**Beamwidth of interest (degrees)**	**Simplified method (%)**	**Manual method (%)**	**Relative error (%)**
1	30	0	120	22	7.0	0.00634	0.00636	–0.322
2	30	0	77	4	5.5	0.0153	0.0154	–0.383
3	35	0	135	25	3.0	0.00099	0.00099	–0.006
4	35	0	82	10	4.5	0.00687	0.00689	–0.255
5	40	0	118	23	4.0	0.00214	0.00214	–0.005
6	40	0	88	23	3.2	0.00148	0.00148	0.198

5 Means of calculating the coordinates of the intersection of two orbital planes

Interference events involving an earth station and two or more satellites frequently occur at the point of closest approach of the two satellites. One case of particular importance concerns Earth observing satellites in sun-synchronous orbits. Their orbits are usually at the same altitude, but their orbital planes are offset. If no consideration has been given to phasing the satellites in their respective orbital planes, it is possible for the satellites to actually cross, one in front of the other. If this occurs while one satellite is being tracked by an earth station, it can result in the earth station being "captured" by the other satellite and starting to track the other satellite. This interference event leads to a loss of data for the period between the loss of lock on the desired satellite and subsequent reacquisition of the desired satellite. The latitude and relative longitude in inertial space where this will occur is relatively easy to calculate.

5.1 Analysis

The geometry of the inertial coordinate system is shown in Fig. 11. There are two orbital planes. The first, which is inclined with respect to the x-y plane by I_1, is offset from the second by $\Delta\lambda_1$. The x-axis lies in the second orbital plane which is inclined by I_2 with respect to the x-y plane. The latitude of the intersection of the two planes is designated by φ_0. The paths of the satellites on the unit orbital sphere are shown by the arcs *a* and *b*. It is well known from spherical trigonometry that there is a simple relationship between the latitude of a location on an inclined circular orbit and the central angle. Thus, for plane No.1:

$$\sin\varphi_1 = \sin b \sin I_1 \tag{42a}$$

and for plane No. 2

$$\sin\varphi_2 = \sin a \sin I_2 \tag{42b}$$

At the point of intersection, $\varphi_1 = \varphi_2$. Therefore:

$$\sin b \sin I_1 = \sin a \sin I_2 \tag{43}$$

Further, from the Law of Sines for oblique spherical triangles:

$$\frac{\sin\Delta\lambda_1}{\sin\gamma} = \frac{\sin a}{\sin I_1} = \frac{\sin b}{\sin(180 - I_2)} \tag{44}$$

Also, from the Law of Cosines for angles:

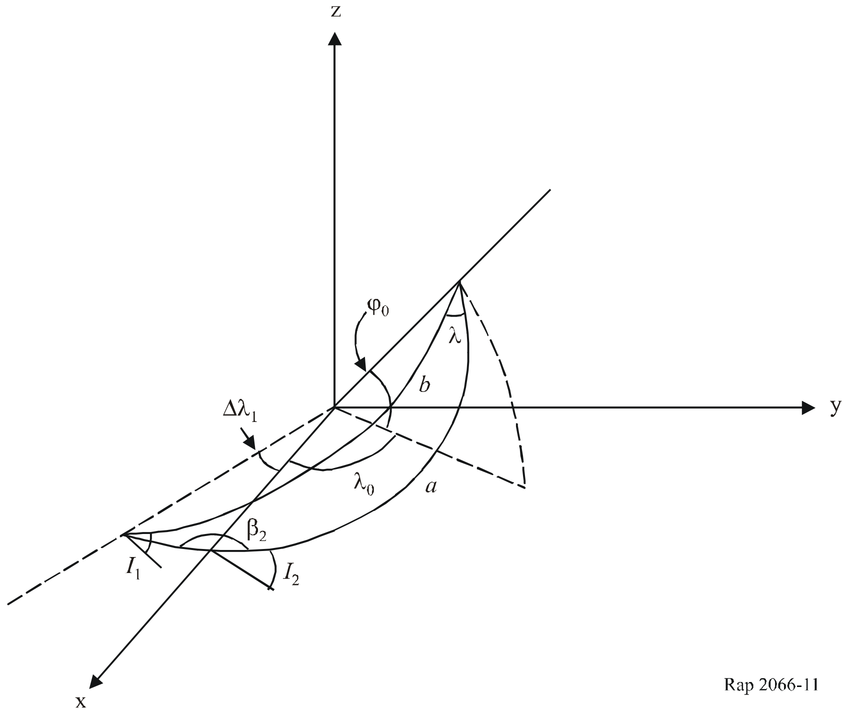
$$\cos\gamma = -\cos I_1 \cos(180 - I_2) + \sin I_1 \sin(180 - I_2)\cos\Delta\lambda_1 \tag{45a}$$

Equation (45a) may be solved for γ:

$$\gamma = \cos^{-1}(\cos I_1 \cos I_2 + \sin I_1 \sin I_2 \cos\Delta\lambda_1) \tag{45b}$$

FIGURE 11

Geometry to determine the latitude and longitude of the intersection of two planes



Consequently, the latitude of intersection is given by:

$$\varphi_0 = \sin^{-1}\left(\sin I_1 \sin I_2 \frac{\sin \Delta\lambda_1}{\sin \gamma}\right) \tag{46}$$

The longitude of intersection is obtained in the following way. From equation (44), the central angle *a* is given by:

$$a = \sin^{-1}\left(\sin I_1 \frac{\sin \Delta\lambda_1}{\sin \gamma}\right) \tag{47a}$$

Further:

$$\lambda_0 = \tan^{-1}\left(\tan \boldsymbol{a} \cos \boldsymbol{I_2}\right) \tag{47b}$$

Table 3 gives several examples of the latitude and longitude of intersection of the orbital planes in an inertial coordinate system. For convenience, the right ascension of the ascending node (RAAN) for satellite No. 2 is assumed to be the x-axis. Note that the latitude and longitude of the intersecting planes is the latitude and longitude of the point where the satellites will cross if the altitude of the satellite orbits are the same.

TABLE 3

Examples of the intersection of offset orbital planes

	Satellite No. 1		Satellite No. 2		Intersection	
Case	**RAAN (degrees)**	**Inclination (degrees)**	**RAAN (degrees)**	**Inclination (degrees)**	**Latitude (degrees)**	**Longitude (degrees)**
1	–5	98.2	0	96.0	65.104	–13.089
2	–5	98.2	0	98.2	81.792	–87.5
3	–10	98.2	0	98.2	81.769	–85.0
4	–15	98.2	0	98.2	81.730	–82.5
5	–20	98.2	0	98.2	81.675	–80.0